CSCE 970 Lecture 5: More Properties of Bayes Nets

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Introduction

- So far, have introduced Bayes nets and discussed the Markov condition

- As mentioned previously, Markov condition entails conditional independencies among variables

- Does not imply any entailed dependencies

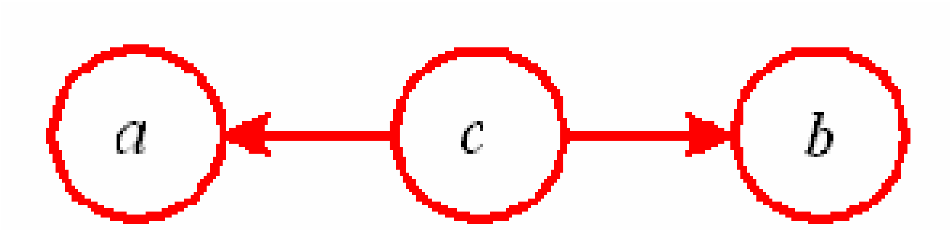
- Throughout lecture, unless otherwise stated, assume that (P, G) satisfies Markov condition

Outline

- Entailed conditional independencies
- Markov equivalence
- Entailing dependencies: faithfulness and embedded faithfulness
- Minimality
- Markov blankets and Markov boundaries

Entailed Conditional Independencies

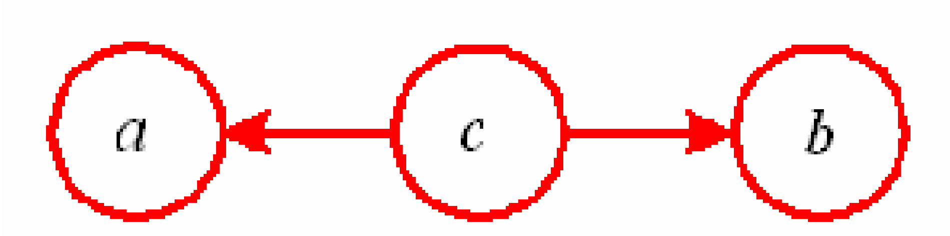
Tail-to-Tail Connections



Are a and b independent? Conditionally independent given c?

Entailed Conditional Independencies

Tail-to-Tail Connections (cont'd)



- Factorization via Theorem 1.4:

$$P(a, b, c) = P(a \mid c)P(b \mid c)P(c)$$

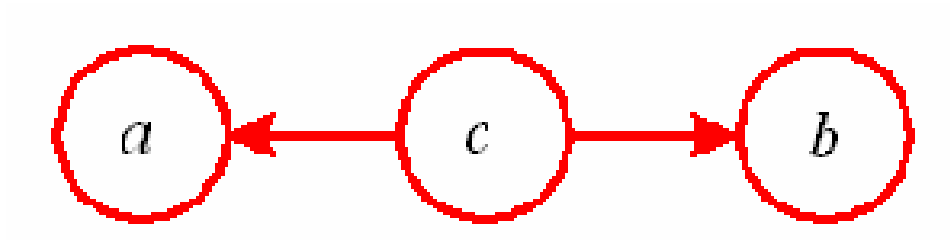
- When c unknown, get $P(a, b)$ by marginalizing:

$$P(a, b) = \sum_{c} P(a \mid c)P(b \mid c)P(c) \ ,$$

which generally does not equal $P(a)P(b)$

Entailed Conditional Independencies

Tail-to-Tail Connections (cont'd)



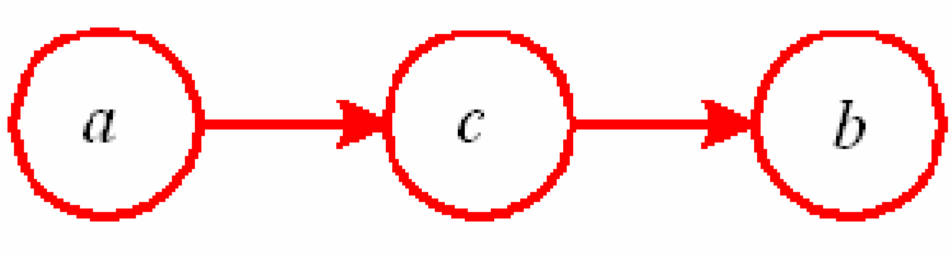
- But when conditioning on c, get:

$$P(a, b \mid c) = \frac{P(a, b, c)}{P(c)} = \frac{P(c)P(a \mid c)P(b \mid c)}{P(c)} = P(a \mid c)P(b \mid c)$$

- Thus a and b conditionally independent given c
- Say that connection between a and b is blocked by c when it is observed and unblocked when unobserved
- Always true for uncoupled tail-to-tail connections $a \leftarrow c \rightarrow b$ (where there's no edge between a and b)

Entailed Conditional Independencies

Head-to-Tail Connections



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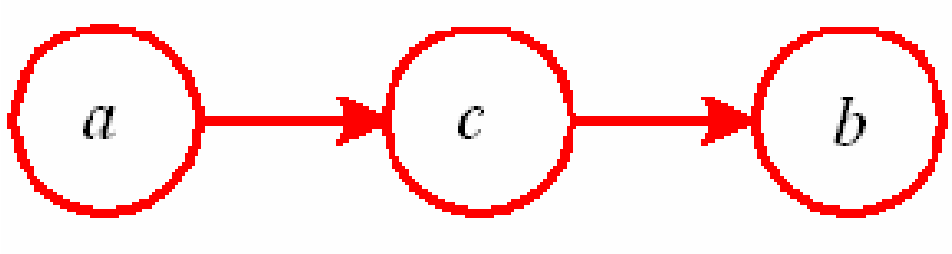
Entailed Conditional Independencies

Head-to-Tail Connections (cont'd)



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- When c unknown, get $P(a, b)$ by marginalizing:

$$P(a, b) = P(a) \sum_c P(c \mid a)P(b \mid c) = P(a)P(b \mid a) \quad ,$$

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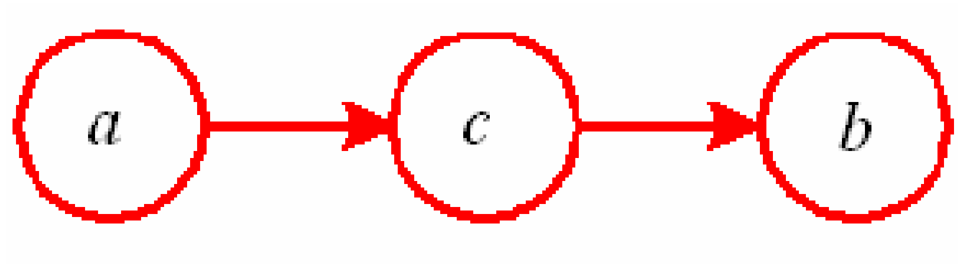
Entailed Conditional Independencies

Head-to-Tail Connections (cont'd)



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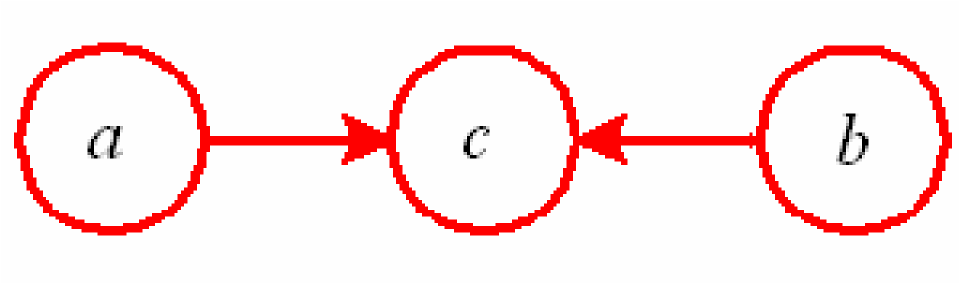
- Always true for uncoupled head-to-tail connections $a \longrightarrow c \longrightarrow b$

Entailed Conditional Independencies

Head-to-Head Connections



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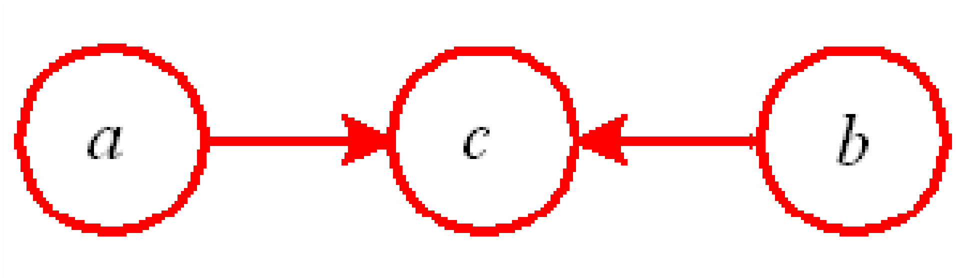
Entailed Conditional Independencies

Head-to-Head Connections (cont'd)



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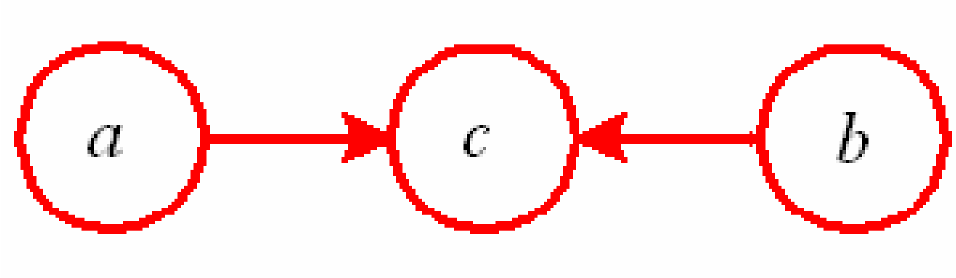
Entailed Conditional Independencies

Head-to-Head Connections (cont'd)



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$$P(a,b \mid c) = \frac{P(a,b,c)}{P(c)} = \frac{P(a)P(b)P(c \mid a,b)}{P(c)} \quad ,$$

  which generally does not equal $P(a \mid c)P(b \mid c)$

- Say that connection between a and b is blocked by c when it is unobserved and unblocked when observed (also unblocks if one of c's descendants is observed)

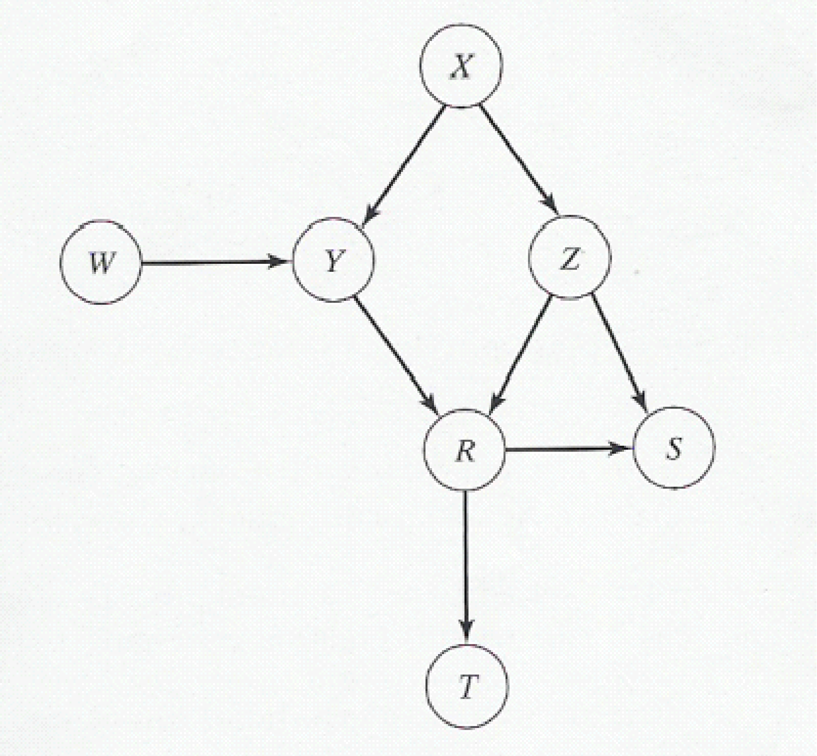
- Always true for uncoupled head-to-head connections $a \rightarrow c \leftarrow b$

D-Separation

- Let a chain of nodes be a sequence of vertices in the DAG G that are pairwise adjacent, ignoring direction of the edges
  - E.g. on the next slide, $[W, Y, X, Z, S, R]$ is a chain
- Two nodes X and Y from G are d-separated by a set of nodes $\mathcal{A} \subset \mathcal{V}$ if every chain from X to Y is blocked by some node in $\mathcal{A}$
- This generalizes to sets of nodes $\mathcal{X}$ and $\mathcal{Y}$ if every pair of nodes (one from $\mathcal{X}$ and one from $\mathcal{Y}$) is d-separated by a node from $\mathcal{A}$
- Theorem 2.1: Based on the Markov condition, a DAG G entails all and only the conditional independencies that are identified by d-separation in G
  - I.e. if (P, G) satisfies the Markov condition, then if one finds a CI in P implied by G, this CI will also be found via d-separation in G
  - Won't necessarily find all CIs in P, since some CIs may not be captured in G

D-Separation

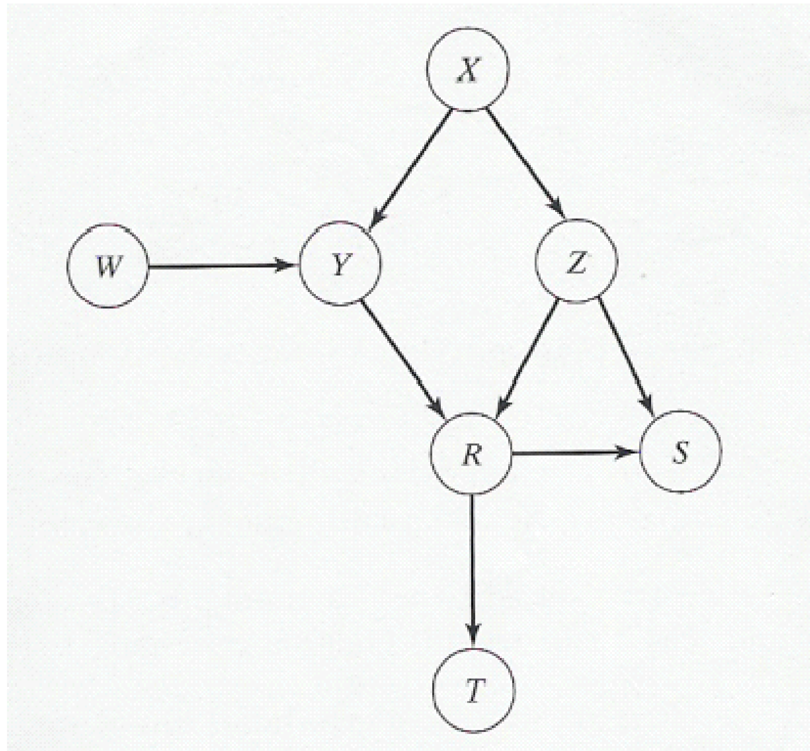
Example



- W and T:
  - Chain $[W, Y, R, T]$ is blocked by Y or R
  - Chain $[W, Y, X, Z, R, T]$ is blocked by X or Z or R
  - Chain $[W, Y, X, Z, S, R, T]$ is blocked by X or Z or R but not by S since observing S unblocks the chain

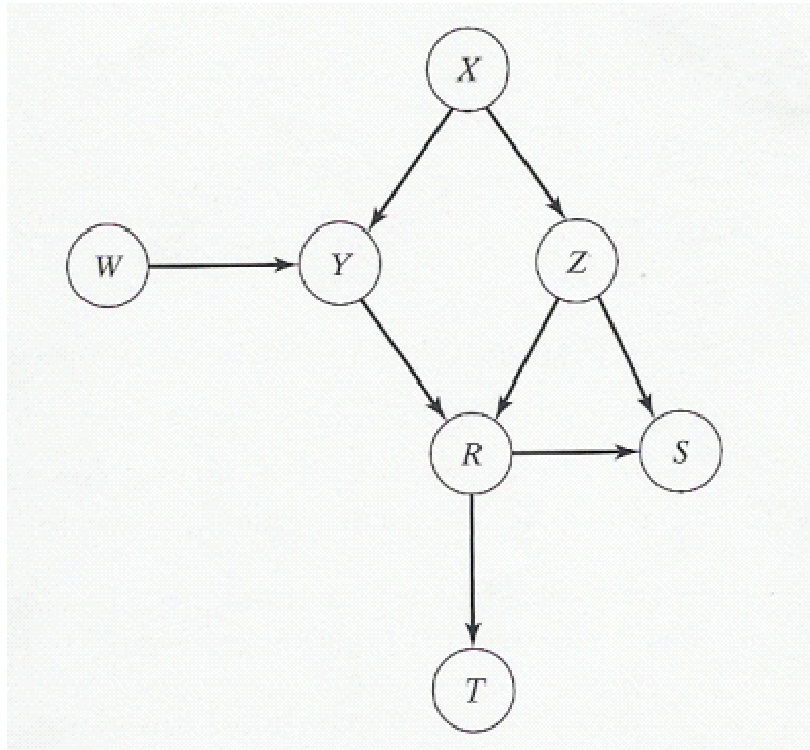
D-Separation

Example (cont'd)



- Y and T:
  - Chain $[Y, R, T]$ is blocked by R
  - Chain $[Y, X, Z, R, T]$ is blocked by X or Z or R
  - Chain $[Y, X, Z, S, R, T]$ is blocked by X or Z or R

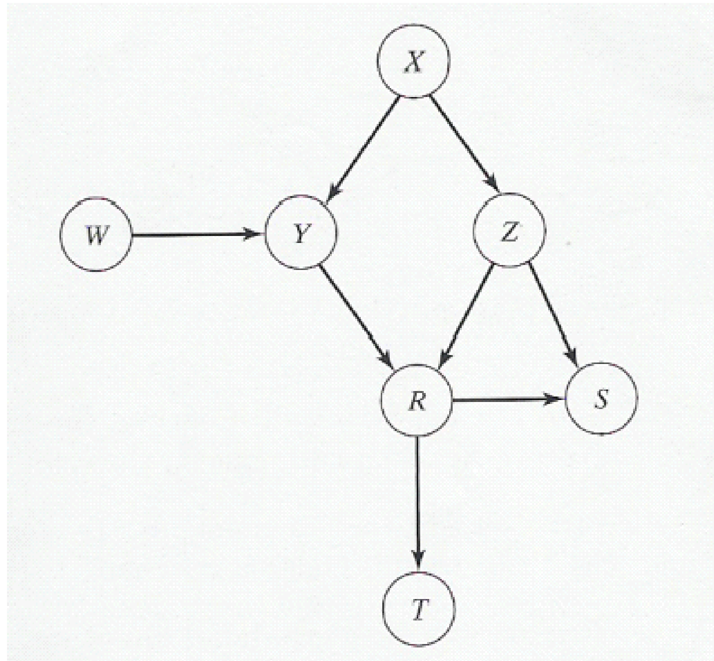
D-Separation
Example (cont'd)



- W and S:
  - Chain $[W, Y, R, S]$ is blocked by Y or R
  - Chain $[W, Y, X, Z, R, S]$ is blocked by X or Z or R
  - Chain $[W, Y, X, Z, S]$ is blocked by X or Z
  - Chain $[W, Y, R, Z, S]$ is blocked by Y or Z

D-Separation

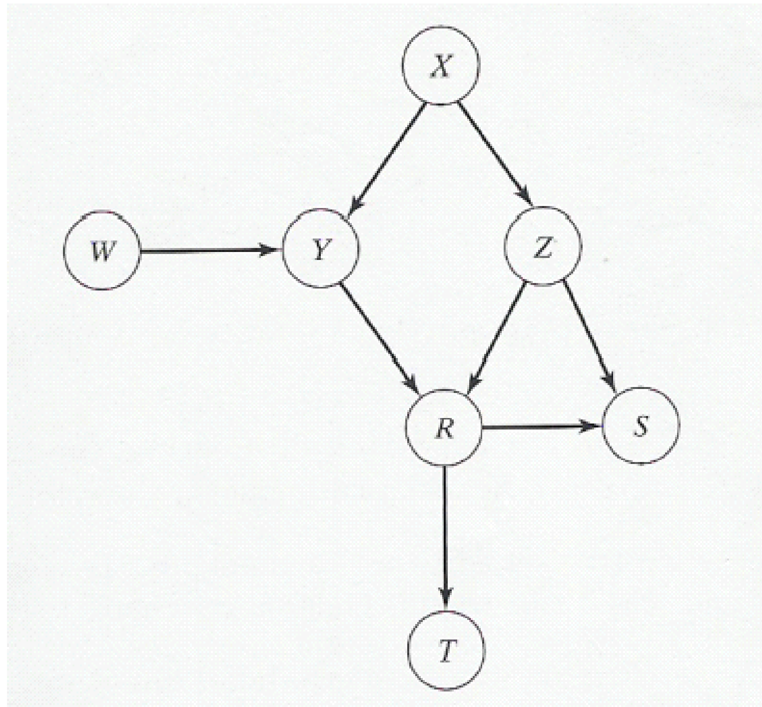
Example (cont'd)



- Y and S:
  - Chain $[Y, R, S]$ is blocked by R
  - Chain $[Y, R, Z, S]$ is blocked by Z
  - Chain $[Y, X, Z, R, S]$ is blocked by X or Z or R
  - Chain $[Y, X, Z, S]$ is blocked by X or Z
- Thus we say that $\{W, Y\}$ and $\{S, T\}$ are conditionally independent given $\{R, Z\}$, i.e. $I_G(\{W, Y\}, \{S, T\} \mid \{R, Z\})$

D-Separation

Another Example



- W and X:
  - Chain $[W, Y, X]$ is blocked by Y when not observed
  - Chain $[W, Y, R, Z, X]$ is blocked by R when not observed
  - Chain $[W, Y, R, S, Z, X]$ is blocked by S when not observed
- Thus we say that W and X are independent, i.e. $I_G(\{W\}, \{X\} \mid \emptyset)$

Finding D-Separations

- Problem: Given a DAG $G = (\mathcal{V}, \mathcal{E})$, and disjoint subsets $\mathcal{A}, \mathcal{B} \subset \mathcal{V}$, find the set of nodes $\mathcal{D}$ that is d-separated from $\mathcal{B}$ by $\mathcal{A}$
  - I.e. find the set of nodes $\mathcal{D}$ that are blocked from those in $\mathcal{B}$ by $\mathcal{A}$
  - I.e. if there is an active path from a node $X \in \mathcal{B}$ to some node $Y \notin \mathcal{A} \cup \mathcal{B}$ (a path from X to Y not blocked by something in $\mathcal{A}$), then Y is NOT in $\mathcal{D}$

- Thus we'll find

$$\mathcal{R} = \{Y : Y \in \mathcal{B} \text{ or } \exists X \in \mathcal{B} \text{ that can reach } Y \text{ with no block from } \mathcal{A}\}$$

(the set of reachable nodes) and set $\mathcal{D} = \mathcal{V} \setminus (\mathcal{A} \cup \mathcal{R})$

Finding D-Separations

(cont'd)

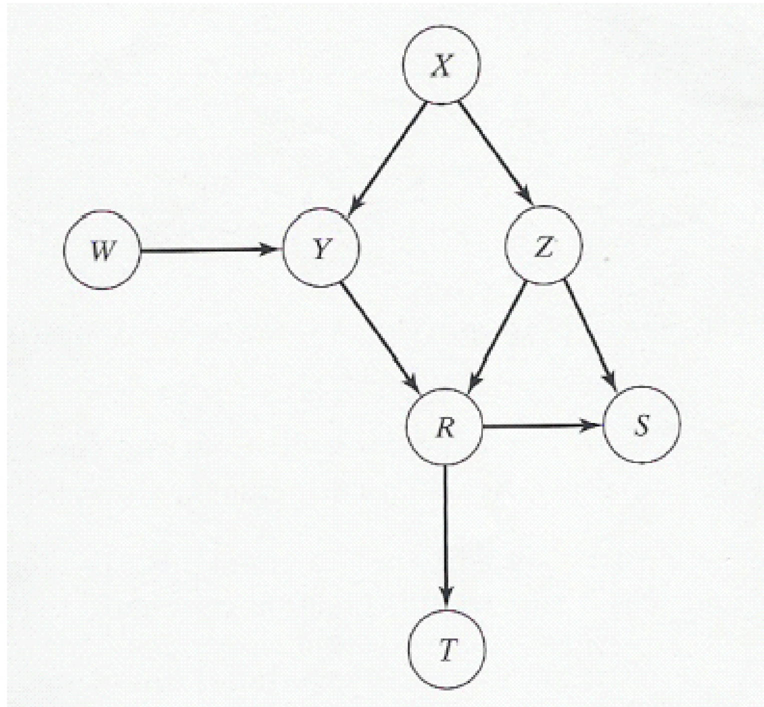
- How does node Z block a chain?
  1. By being in a head-to-tail or tail-to-tail arrangement in the chain and being in $\mathcal{A}$

     OR
  2. By being in a head-to-head arrangement in the chain not being in $\mathcal{A}$ and not having a descendent in $\mathcal{A}$

- Since we're initially seeking (sort of) the complement of $\mathcal{D}$, we'll turn the above two conditions on their heads and look for a set of nodes $\mathcal{R}$ that are reachable from $\mathcal{B}$ via active chains

- A chain is active iff each of its 3-node subchains $U - V - W$ satisfies one of
  1. $U - V - W$ is not head-to-head at V and $V \notin \mathcal{A}$
  2. $U - V - W$ is head-to-head at V and $V \in \mathcal{A}$ or a descendent of V is in $\mathcal{A}$

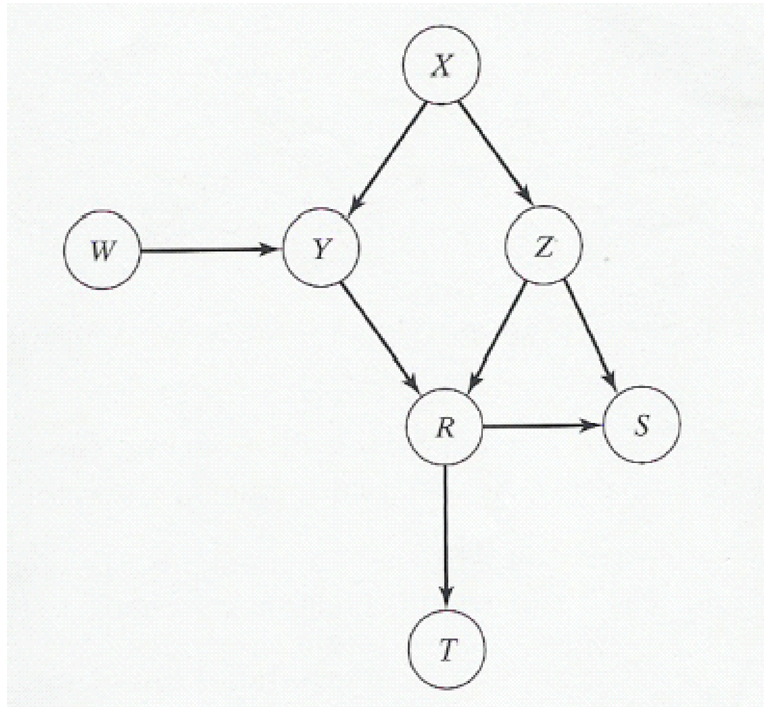
Finding D-Separations (cont'd)



- Let $\mathcal{B} = \{W, Y\}$ and $\mathcal{A} = \{X\}$
  - Then the active chains out of nodes in $\mathcal{B}$ are $[Y, R, T]$, $[Y, R, S]$, $[W, Y, R, T]$, $[W, Y, R, S]$, and $[W, Y, R]$

$\Rightarrow$ D-separation from $\{Z\}$

Finding D-Separations (cont'd)



- Let $\mathcal{B} = \{W, Y\}$ and $\mathcal{A} = \{X, T\}$
  - Then the active chains out of nodes in $\mathcal{B}$ are $[Y, R, Z]$, $[Y, R, S]$, $[Y, R, Z, S]$, $[W, Y, R]$, $[W, Y, R, Z]$, $[W, Y, R, S]$, and $[W, Y, R, Z, S]$

$\Rightarrow$ D-separation from $\emptyset$

Finding D-Separations

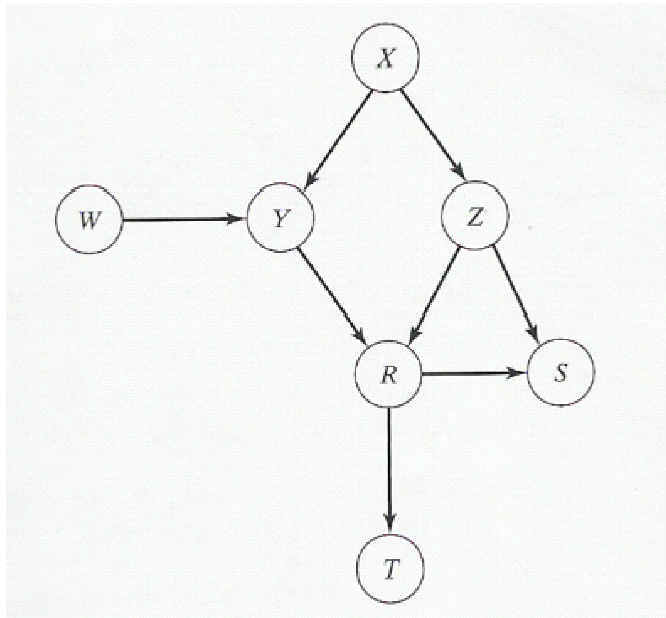
(cont'd)

- This problem is a node reachability problem with restrictions to legal pairs of edges

- Define a pair of edges $((U, V), (V, W))$ to be legal iff they satisfy one of the two active chain conditions described earlier

- Then $\mathcal{R}$ is the set of nodes reachable from a node in $\mathcal{B}$ via only legal pairs of edges

Finding D-Separations (cont'd)



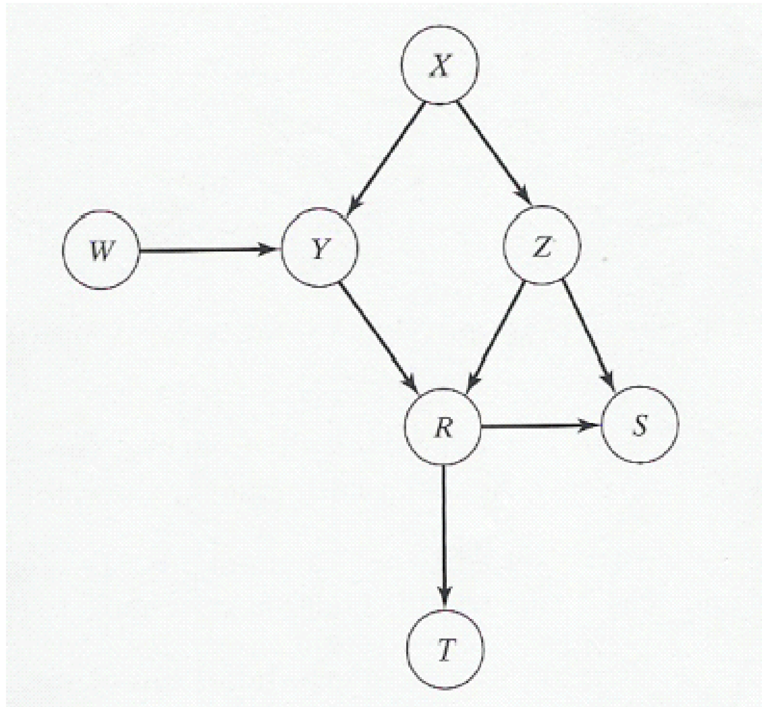
- Let $\mathcal{B} = \{W, Y\}$ and $\mathcal{A} = \{X\}$
  - Then the set of legal pairs of edges is (excluding symmetries)

$$\mathcal{L} = \{((X, Z), (Z, R)), ((X, Z), (Z, S)), ((X, Y), (Y, R)),$$

$$((W, Y), (Y, R)), ((Y, R), (R, T)), ((Y, R), (R, S)),$$

$$((Z, R), (R, T)), ((Z, R), (R, S)), ((R, Z), (Z, S))\}$$

Finding D-Separations (cont'd)



- Let $\mathcal{B} = \{W, Y\}$ and $\mathcal{A} = \{X, T\}$
  - Then the set of legal pairs of edges is (excluding symmetries) the same as before, but add $((Y, R), (R, Z))$ and $((W, Y), (Y, X))$ (why?)

Finding D-Separations

The Algorithm

1. Given $G = (\mathcal{V}, \mathcal{E})$, $\mathcal{B}$, and $\mathcal{A}$, compute the set of legal edge pairs $\mathcal{L}$

2. Create $G' = (\mathcal{V}, \mathcal{E}')$, which is G with opposite edges added:

$$\mathcal{E}' = \mathcal{E} \cup \{(X, Y) : (Y, X) \in \mathcal{E}\}$$

   - Because the reachability algorithm respects edges' directions, but d-separation does not

3. Run as a subroutine an algorithm to return $\mathcal{R}$, the set of nodes in G' that are reachable from $\mathcal{B}$ via edge pairs from $\mathcal{L}$

4. The set of nodes that are d-separated from $\mathcal{B}$ by $\mathcal{A}$ is $\mathcal{D} = \mathcal{V} \setminus (\mathcal{A} \cup \mathcal{R})$

Finding D-Separations

Reachability Subroutine

- A breadth-first search of graph G', but over edges rather than nodes

1. Initialize $i = 1$ and

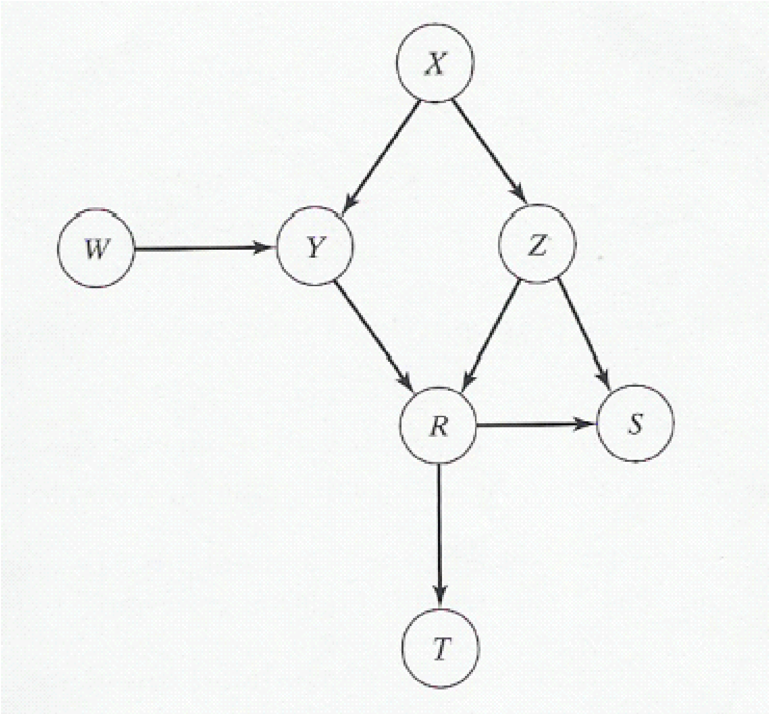
$$\mathcal{R} = \mathcal{B} \cup \{V : V \in \mathcal{V} \text{ and } (X, V) \in \mathcal{E}' \text{ for some } X \in \mathcal{B}\}$$

2. Label each such edge (X, V) with a 1

3. While new nodes added to $\mathcal{R}$
   (a) For each V such that edge (U, V) is labeled i
       i. For each unlabeled edge (V, W) s.t. $((U, V), (V, W)) \in \mathcal{L}$
          A. $\mathcal{R} = \mathcal{R} \cup \{W\}$
          B. Label (V, W) with $i + 1$
   (b) $i + +$

Finding D-Separations

Team Exercise



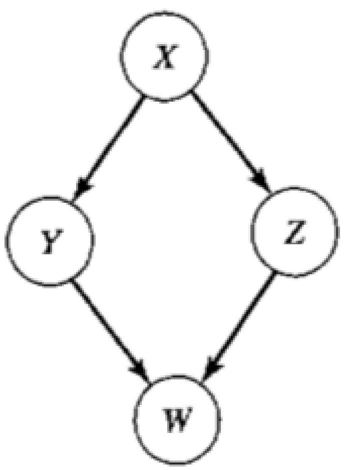
- Let $\mathcal{B} = \{W, Y\}$ and $\mathcal{A} = \{X\}$

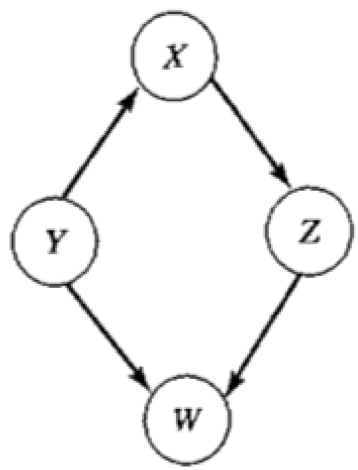
- Everybody join one of four teams (even if you're just sitting in), draw this graph, and simulate the algorithm, including labeling edges

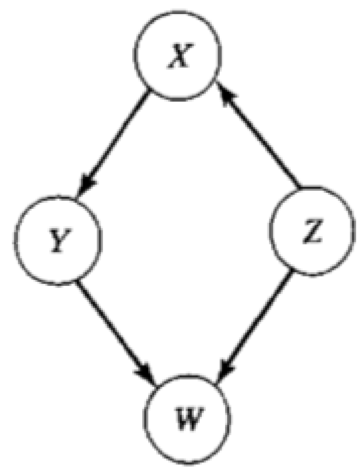
Markov Equivalence

- Many DAGs with the same set of vertices have the same d-separations

- DAGs $G_1 = (\mathcal{V}, \mathcal{E}_1)$ and $G_2 = (\mathcal{V}, \mathcal{E}_2)$ are Markov equivalent if for every three mutually disjoint subsets $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \mathcal{V}$, $\mathcal{A}$ and $\mathcal{B}$ are d-separated by $\mathcal{C}$ in G_1 iff $\mathcal{A}$ and $\mathcal{B}$ are d-separated by $\mathcal{C}$ in G_2
  - I.e. $I_{G_1}(\mathcal{A}, \mathcal{B} \mid \mathcal{C}) \Leftrightarrow I_{G_2}(\mathcal{A}, \mathcal{B} \mid \mathcal{C})$





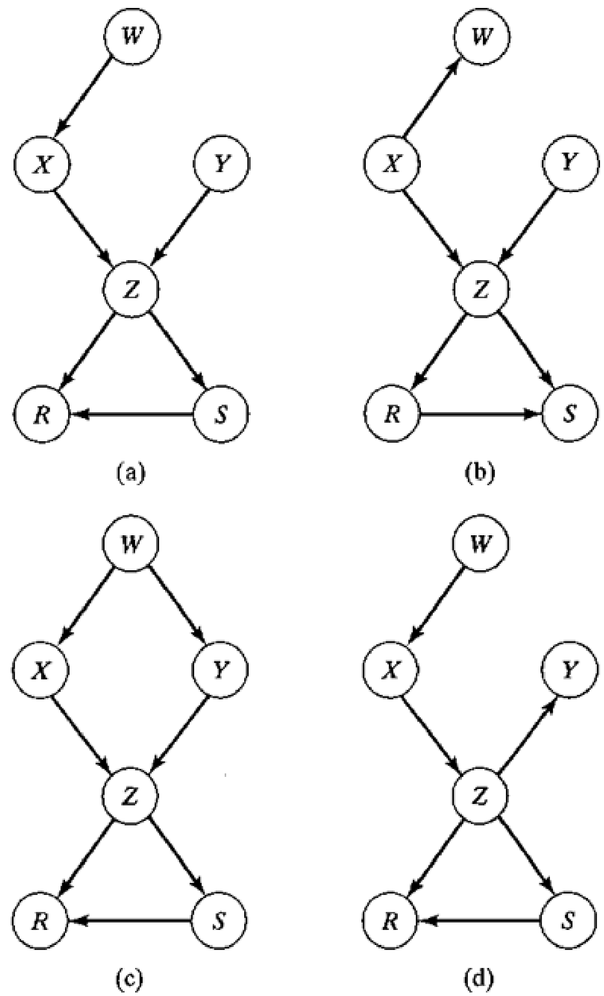


Markov Equivalence

(cont'd)

Theorem 2.4: DAGs G_1 and G_2 are Markov equivalent iff they have the same links (ignoring edge direction) and the same set of uncoupled head-to-head matchings

Markov Equivalence
(cont'd)

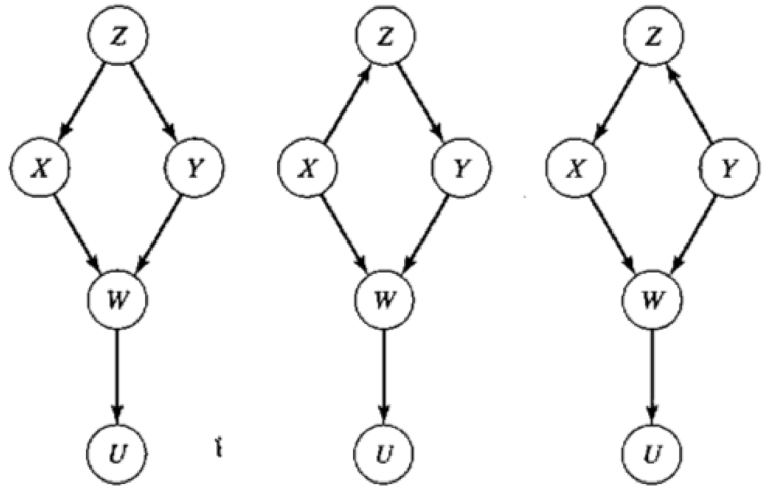


DAG Patterns

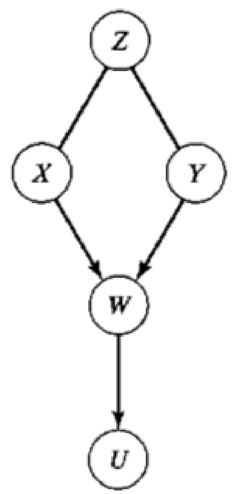
- Can represent a set of Markov equivalent DAGs in a single graph
- If an edge can be directed either way and still yield a Markov equivalent DAG, then the edge in the DAG pattern is undirected
- If the edge must be oriented only one way, then the edge in the DAG pattern remains directed

DAG Patterns

(cont'd)

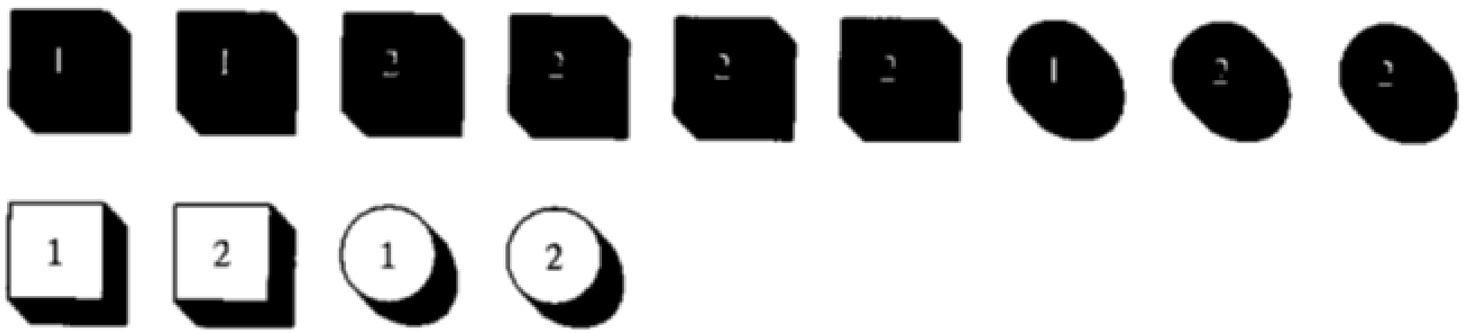


(a)



(b)

Entailing Dependencies

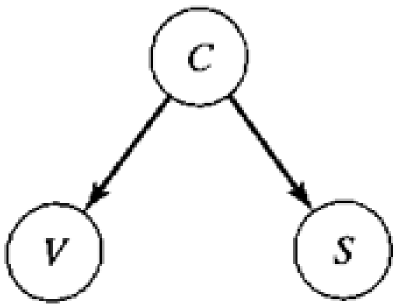


P is uniform

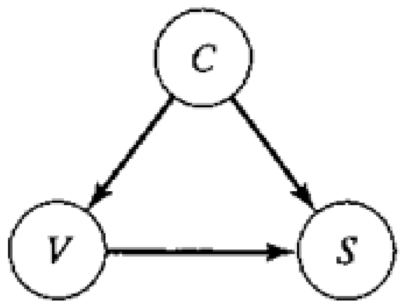
Var	Values	Outcomes
V	$\{v1, v2\}$	obj with "1"/"2"
S	$\{s1, s2\}$	square/round
C	$\{c1, c2\}$	black/white

Entailing Dependencies (cont'd)

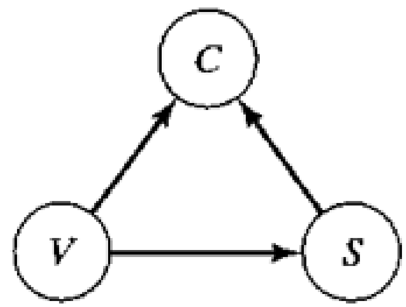
We earlier showed that $I_P(\{V\},\{S\} \mid \{C\})$. All of the following three graphs have the Markov property with P.



(a)



(b)



(c)

Graphs (b) and (c) have no independencies, so they satisfy the Markov condition with <u>any</u> distribution P

Entailing Dependencies

Faithfulness

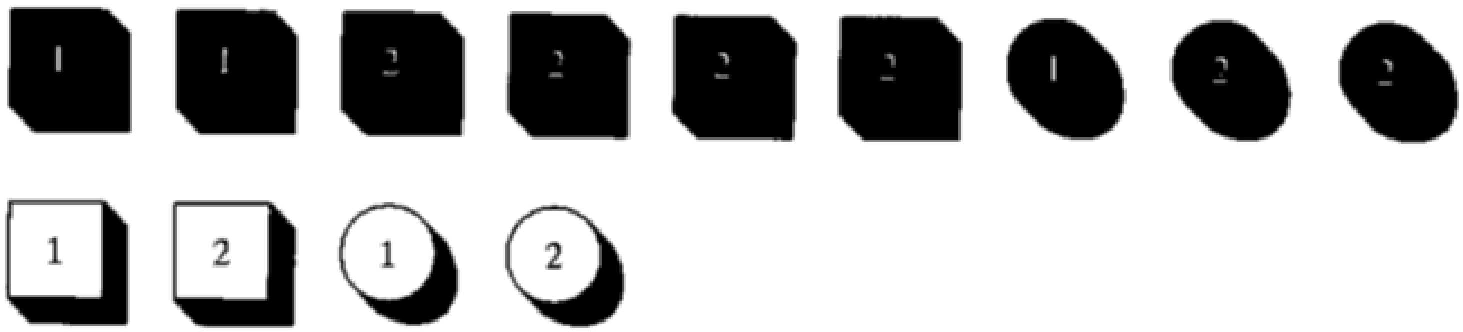
- Given a DAG G and a distribution P, (G, P) satisfies the faithfulness condition if both of these conditions hold

1. (G, P) satisfies the Markov condition

2. All conditional independencies in P are entailed by G, based on the Markov condition

Entailing Dependencies

Faithfulness Example



P is uniform

Var	Values	Outcomes
V	$\{v1, v2\}$	obj with "1"/"2"
S	$\{s1, s2\}$	square/round
C	$\{c1, c2\}$	black/white

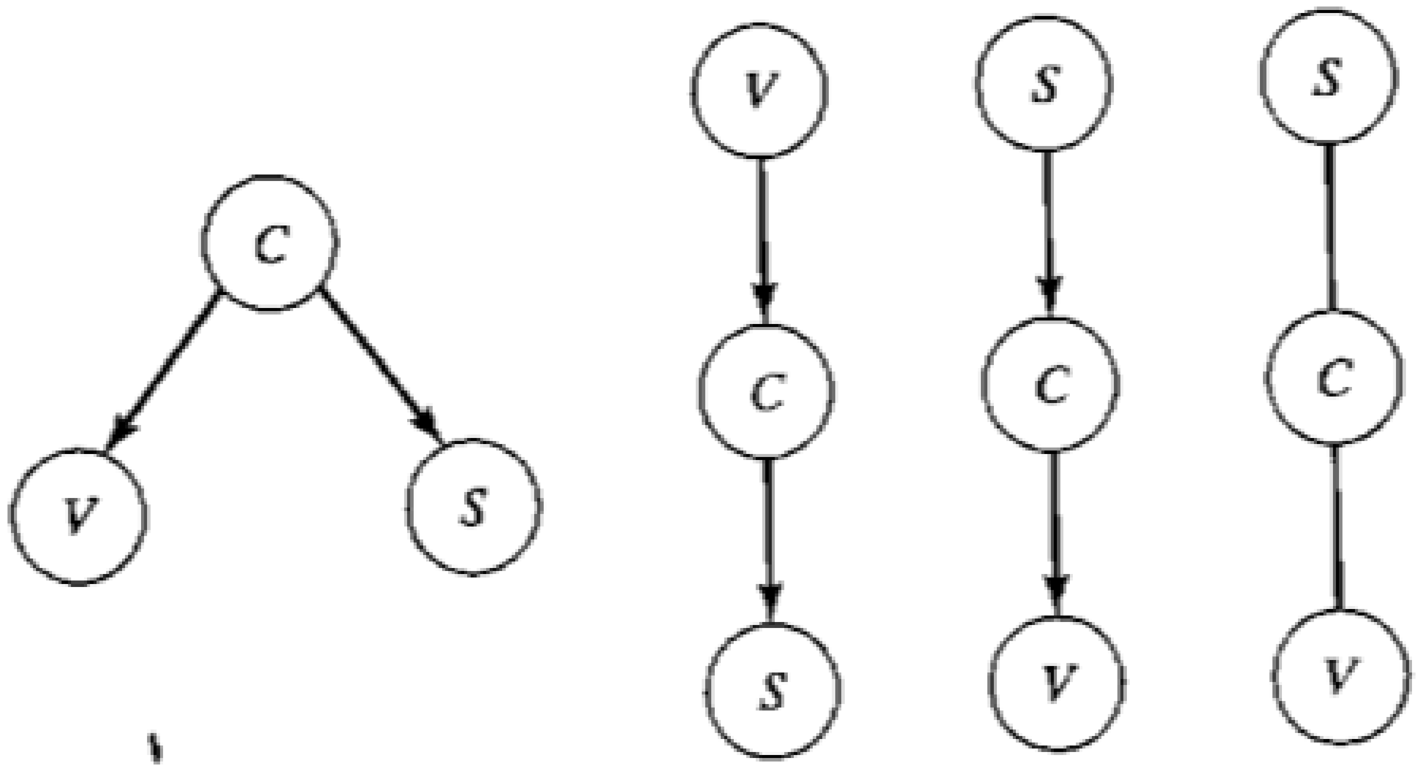
c	s	v	$P(v)$	$P(s)$	$P(v, s)$
$c1$	$s1$	$v1$	$5/13$	$8/13$	$3/13$
$c1$	$s1$	$v2$	$8/13$	$5/13$	$5/13$
$c1$	$s2$	$v1$	$5/13$	$8/13$	$2/13$
$c1$	$s2$	$v2$	$8/13$	$5/13$	$3/13$
$c2$	$s1$	$v1$	$5/13$	$8/13$	$3/13$
$c2$	$s1$	$v2$	$8/13$	$5/13$	$5/13$
$c2$	$s2$	$v1$	$5/13$	$8/13$	$2/13$
$c2$	$s2$	$v2$	$8/13$	$5/13$	$3/13$

$\Rightarrow \neg I_P(\{V\}, \{S\})$. Can show P's only CI is $I_P(\{V\}, \{S\} \mid \{C\})$

Entailing Dependencies

Faithfulness Example (cont'd)

These are all faithful to P



(a) (b) (c) (d)

Entailing Dependencies

Another Faithfulness Example

$P(x1) = a$	$P(y1\|x1) = 1 - (b + c)$	$P(z1\|y1) = e$
$P(x2) = 1 - a$	$P(y2\|x1) = c$	$P(z2\|y1) = 1 - e$
	$P(y3\|x1) = b$	
		$P(z1\|y2) = e$
	$P(y1\|x2) = 1 - (b + d)$	$P(z2\|y2) = 1 - e$
	$P(y2\|x2) = d$	
	$P(y3\|x2) = b$	$P(z1\|y3) = f$
		$P(z2\|y3) = 1 - f$

G does not entail unconditional independence of X and Z, but P does

$\Rightarrow$ Markov property holds, but P not faithful to G

Entailing Dependencies

Another Faithfulness Example (cont'd)

Turns out that $P(X,Z) = P(X)P(Z)$. E.g.

$$\begin{aligned} P(y3) &= \sum_x P(y3 \mid x)P(x) = ba + b(1-a) = b \\ P(y2) &= \sum_x P(y1 \mid x)P(x) = ca + d(1-a) = ca + d - da \\ P(y1) &= (1-(b+c))a + (1-(b+d))(1-a) = 1 - ac + ad - b - d \end{aligned}$$

$$\begin{aligned} P(z1) &= e(1 - ac + ad - b - d) + e(ca + d - da) + fb \\ &= e - eb + fb \end{aligned}$$

$$\Rightarrow P(x1)P(z1) = a(e - eb + fb)$$

$$\begin{aligned} P(z1, x1) &= P(z1 \mid x1)P(x1) = P(x1)\sum_y P(z1 \mid y)P(y \mid x1) \\ &= a[e(1-(b+c)) + ec + fb] = a(e - eb + fb) \end{aligned}$$

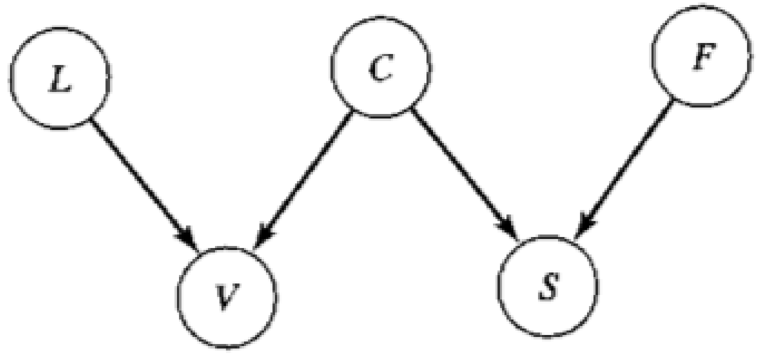
Faithful DAG Representations

- Theorem 2.6: If (G, P) satisfies the faithfulness condition, then P satisfies this with all and only those DAGs that are Markov equivalent with G

- The graph pattern representing the class of Markov equivalent DAGs that P is faithful to is called a perfect map of P

- P admits a faithful DAG representation if it is faithful to some DAG
  - Not all distributions admit a faithful DAG representation

Faithful DAG Representations (cont'd)



- Consider a joint distribution $P(v, s, c, \ell, f)$ faithful to the above DAG G

- Only independencies (excluding those with C) are $I_P(\{L\}, \{F, S\})$, $I_P(\{L\}, \{S\})$, $I_P(\{L\}, \{F\})$, $I_P(\{F\}, \{L, V\})$, $I_P(\{F\}, \{V\})$

- Now consider marginal distribution $P(v, s, \ell, f)$. If the marginal is faithful to a DAG G', then the above independencies imply G''s only d-separations

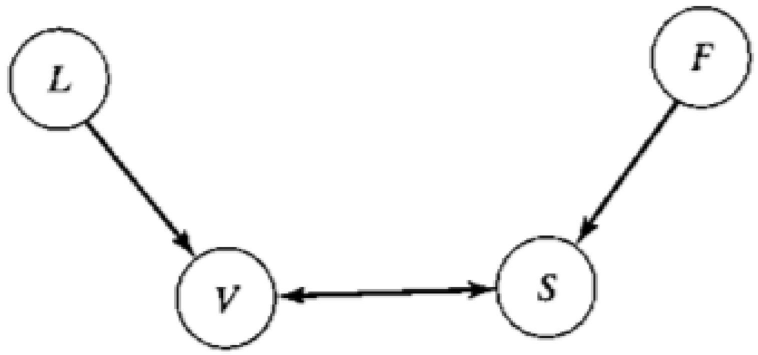
Faithful DAG Representations (cont'd)

- If two nodes cannot be d-separated, then they must be adjacent (Lemma 2.4), so G' has links $L - V$, $V - S$, and $S - F$

- Since $I_{G'}(\{L\}, \{S\})$, the uncoupled meeting $L - V - S$ must be head-to-head

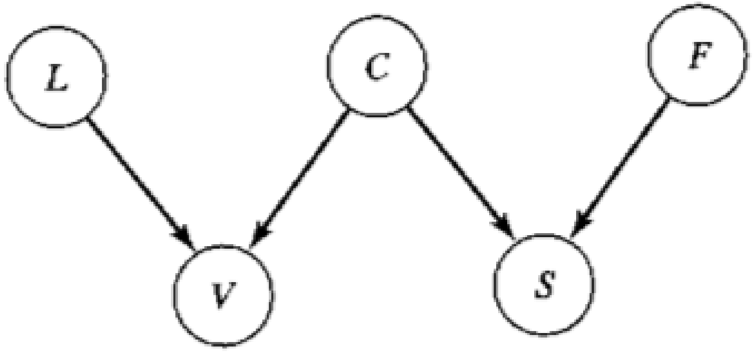
- Also, since $I_{G'}(\{V\}, \{F\})$, the uncoupled meeting $V - S - F$ must be head-to-head

Faithful DAG Representations (cont'd)



Thus G' doesn't exist as a DAG, and the marginal $P(v, s, \ell, f)$ does not admit a faithful DAG representation

Embedded Faithfulness



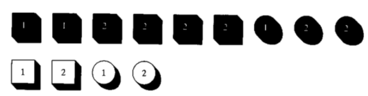
- So $P(v, s, \ell, f)$ does not admit a faithful DAG representation, but if we allow node C to exist as well, then everything works
- Let P be a distribution over $\mathcal{V} \subseteq \mathcal{W}$ and let $G = (\mathcal{V}, \mathcal{E})$ be a DAG. (G, P) satisfies the embedded faithfulness condition if

  1. The CIs entailed by G (when restricting to nodes in V) all exist in P

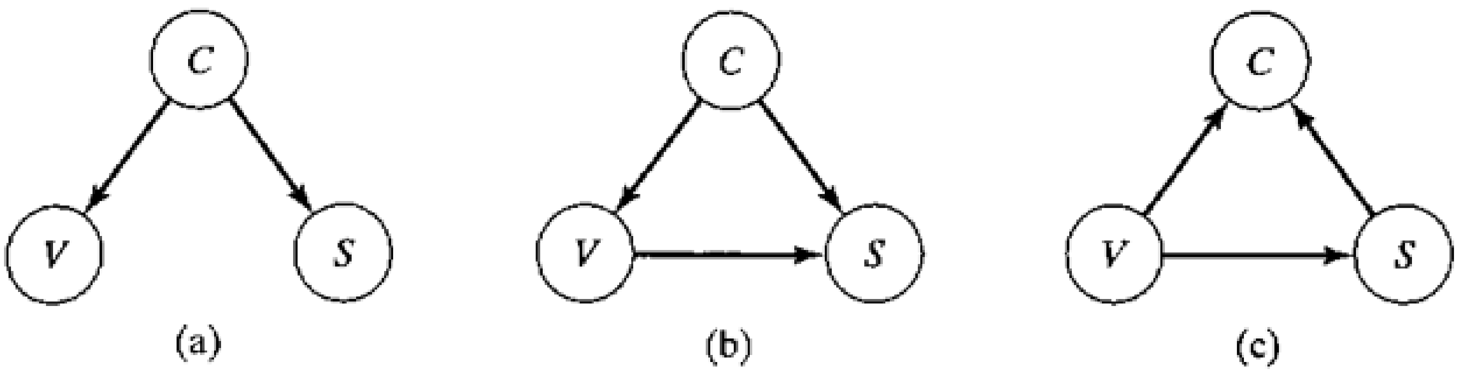
  2. All CIs in P are entailed by G

- P also embedded faithfully in DAG G' that is Markov equivalent to G (and possibly others)

Minimality

Here's that distribution again:  , etc.

The only CI is $I_P(\{V\}, \{S\} \mid C)$, so these have the Markov property:



- If we remove edge (V, S) from (b), it still has the Markov property
- Can we remove any edge from (a) or (c) and still satisfy Markov?
- Given distribution P and DAG $G = (\mathcal{V}, \mathcal{E})$, (G, P) satisfies the minimality condition if (1) (G, P) satisfies the Markov condition and (2) removing any edge from G results in a graph that does not
- Faithfulness $\Rightarrow$ Minimality, but Minimality $\not\Rightarrow$ Faithfulness

Markov Blankets and Boundaries

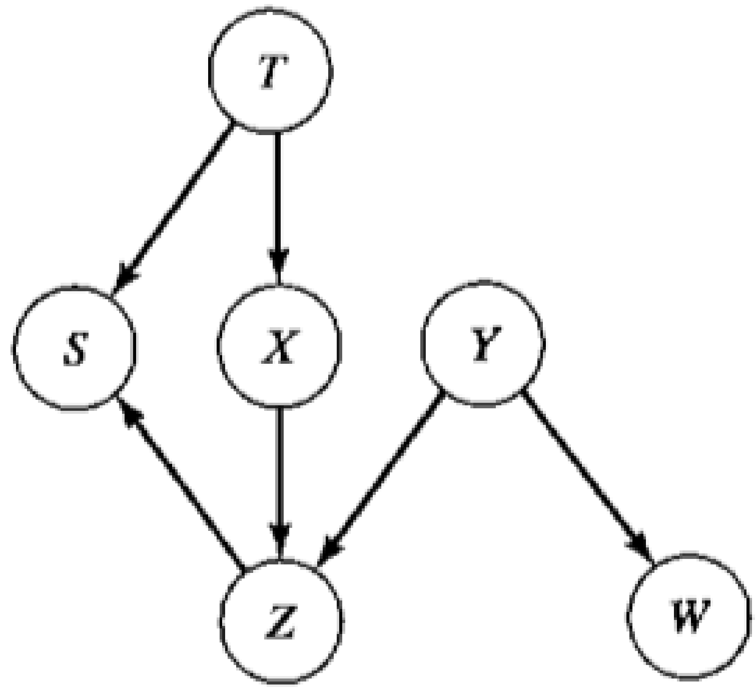
- Let $\mathcal{V}$ be a set of RVs, P their joint distribution, and $X \in \mathcal{V}$. A Markov blanket $\mathcal{M}_X$ of X is any set of variables such that X is CI of all other variables given $\mathcal{M}_X$:

$$I_P(\{X\}, \mathcal{V} \setminus (\mathcal{M}_X \cup \{X\}) \mid \mathcal{M}_X)$$

- If no proper subset of $\mathcal{M}_X$ is a Markov blanket, then $\mathcal{M}_X$ is a Markov boundary
- Theorem 2.13: If (G, P) satisfies the Markov condition, then the set of X's parents, children, and co-parents (other parents of X's children) form a Markov blanket of X
  - "Parent" respects edge direction
- Theorem 2.14: If (G, P) satisfies the faithfulness condition, then the set of X's parents, children, and co-parents form the unique Markov boundary of X

Markov Blankets and Boundaries

Example



- If the faithfulness condition is satisfied, then what is X's Markov boundary?

- What if the edge (T, X) is deleted?