Computer Science & Engineering 423/823
Design and Analysis of Algorithms

Lecture 09 — Lower Bounds (Sections 8.1 and 33.3)

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Remember when ...

... I said: "Upper Bound of an Algorithm"

- An algorithm A has an **upper bound** of $f(n)$ for input of size n if there exists **no input** of size n such that A requires more than $f(n)$ time
- E.g., we know from prior courses that Quicksort and Bubblesort take no more time than $O(n^2)$, while Mergesort has an upper bound of $O(n \log n)$

... I said: "Upper Bound of a Problem"

- A problem has an **upper bound** of $f(n)$ if there exists **at least one** algorithm that has an upper bound of $f(n)$
  - I.e., there exists an algorithm with time/space complexity of at most $f(n)$ on **all** inputs of size n
- E.g., since **algorithm** Mergesort has worst-case time complexity of $O(n \log n)$, the **problem** of sorting has an upper bound of $O(n \log n)$

Remember when ...

... I said: "Lower Bound of a Problem"

- A problem has a **lower bound** of $f(n)$ if, for **any** algorithm A to solve the problem, there exists **at least one** input of size n that forces A to take at least $f(n)$ time/space
- This pathological input depends on the specific algorithm A
- E.g., reverse order forces Bubblesort to take $\Omega(n^2)$ steps
- Since **every** sorting algorithm has an input of size n forcing $\Omega(n \log n)$ steps, sorting problem has **time complexity lower bound** of $\Omega(n \log n)$
- To argue a lower bound for a problem, can use an **adversarial** argument: An algorithm that simulates **arbitrary** algorithm A to build a pathological input
  - Needs to be in some general (algorithmic) form since the nature of the pathological input depends on the specific algorithm A
  - Adversary has unlimited computing resources
- Can also **reduce** one problem to another to establish lower bounds

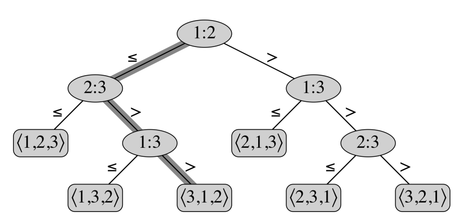
Comparison-Based Sorting Algorithms

- Our lower bound applies only to **comparison-based sorting algorithms**
  - The sorted order it determines is based **only** on comparisons between the input elements
  - E.g., Insertion Sort, Selection Sort, Mergesort, Quicksort, Heapsort
- What is **not** a comparison-based sorting algorithm?
  - The sorted order it determines is based on additional information, e.g., bounds on the range of input values
  - E.g., Counting Sort, Radix Sort

Decision Trees

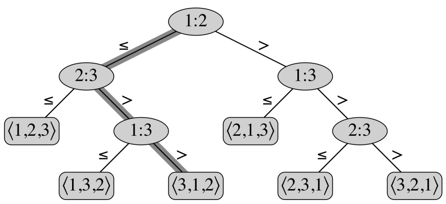
- A **decision tree** is a full binary tree that represents comparisions between elements performed by a particular sorting algorithm operating on a certain-sized input (n elements)
- **Key point:** a tree represents an algorithm's behavior on *all possible inputs* of size n
  - Thus, an adversarial argument could use such a tree to choose a pathological input
- Each internal node represents one comparison made by algorithm
  - Each node labeled as $i : j$, which represents comparison $A[i] \leq A[j]$
  - If, in the particular input, it is the case that $A[i] \leq A[j]$, then control flow moves to left child, otherwise to the right child
  - Each leaf represents a possible output of the algorithm, which is a permutation of the input
  - All permutations must be in the tree in order for algorithm to work properly

Example for Insertion Sort



- If $n = 3$, Insertion Sort first compares $A[1]$ to $A[2]$
- If $A[1] \leq A[2]$, then compare $A[2]$ to $A[3]$
- If $A[2] > A[3]$, then compare $A[1]$ to $A[3]$
- If $A[1] \leq A[3]$, then sorted order is $A[1], A[3], A[2]$

Example for Insertion Sort (2)



- Example: $A = [7, 8, 4]$
- First compare 7 to 8, then 8 to 4, then 7 to 4
- Output permutation is $\langle 3, 1, 2\rangle$, which implies sorted order is 4, 7, 8
- What are worst-case inputs for this algorithm? What are not?

Proof of Lower Bound

- Length of path from root to output leaf is number of comparisons made by algorithm on that input
- Worst-case number of comparisons = length of longest path = **height** h

$\Rightarrow$ Adversary chooses a deepest leaf to create worst-case input

- Number of leaves in tree is $n!$ = number of outputs (permutations)
- A binary tree of height h has at most 2^h leaves
- Thus we have $2^h \geq n! \geq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$
- Take base-2 logs of both sides to get

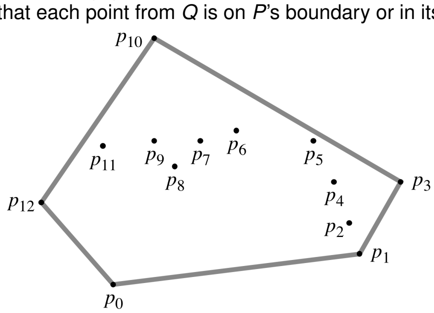
$$h \geq \lg \sqrt{2\pi} + (1/2)\lg n + n \lg n - n \lg e = \Omega(n \log n)$$

$\Rightarrow$ **Every** comparison-based sorting algorithm has **some** input that forces it to make $\Omega(n \log n)$ comparisons $\square$

$\Rightarrow$ Mergesort and Heapsort are *asymptotically optimal*

Another Lower Bound: Convex Hull

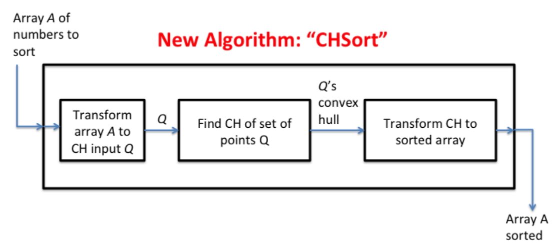
- Use sorting lower bound to get lower bound on **convex hull** problem:
  - Given a set $Q = \{p_1, p_2, \ldots, p_n\}$ of n points, each from $\mathbb{R}^2$, output CH(Q), which is the smallest convex polygon P such that each point from Q is on P's boundary or in its interior



Example output of CH algorithm: ordered set $\langle p_{10}, p_3, p_1, p_0, p_{12}\rangle$

Another Lower Bound: Convex Hull (2)

- **Reduce** problem of sorting to that of finding convex hull
- I.e., given any instance of the sorting problem $A = \{x_1, \ldots, x_n\}$, we will transform it to an instance of convex hull such that the time complexity of the new algorithm sorting will be no more than that of convex hull

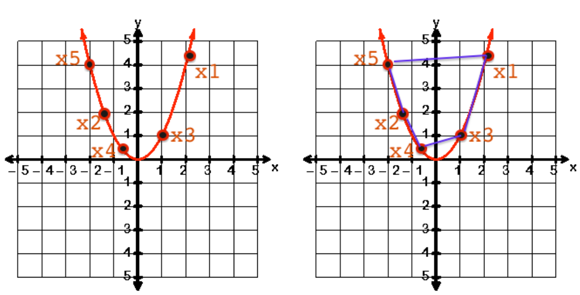


- Reduction: transform A to $Q = \{(x_1, x_1^2), (x_2, x_2^2), \ldots, (x_n, x_n^2)\}$

$\Rightarrow$ Takes $O(n)$ time

Another Lower Bound: Convex Hull (3)

E.g., $A = \{2.1, -1.4, 1.0, -0.7, -2.0\}$,
CH(Q) $= \langle(-1.4, 1.96), (-2, 4), (2.1, 4.41), (1, 1), (-0.7, 0.49)\rangle$



- Since the points in Q are on a parabola, all points of Q are on CH(Q)
- How can we get a sorted version of A from this?

Another Lower Bound: Convex Hull (4)

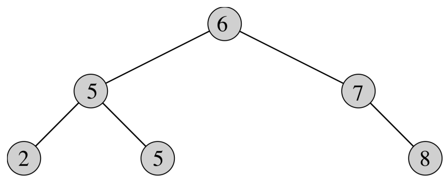
- CHSort yields a sorted list of points from (**any**) A
- Time complexity of CHSort: time to transform A to Q + time to find CH of Q + time to read sorted list from CH

$\Rightarrow$ $O(n)$+ time to find CH $+O(n)$

- If time for convex hull is $o(n \log n)$, then sorting is $o(n \log n)$
  - $\Rightarrow$ Since that cannot happen, we know that convex hull is $\Omega(n \log n)$

In-Class Team Exercise

- A **binary search tree** (BST) has a key value at each node
- For any node x in the tree, the key values of all nodes in x's left subtree are $\leq x$, and the key values of all nodes in x's right subtree are $\geq x$



- Prove that, given an unsorted array A of n elements, the time required to build a BST is $\Omega(n \log n)$ in the worst case