Computer Science & Engineering 423/823 Design and Analysis of Algorithms

Lecture 04 — Greedy Algorithms (Chapter 16)

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Introduction

- Greedy methods: A technique for solving **optimization problems**
  - Choose a solution to a problem that is best per an objective function
- Similar to dynamic programming in that we examine subproblems, exploiting optimal substructure property
- Key difference: In dynamic programming we considered **all** possible subproblems
- In contrast, a greedy algorithm at each step commits to just one subproblem, which results in its **greedy choice** (locally optimal choice)
- Examples: Minimum spanning tree, single-source shortest paths

Activity Selection (1)

- Consider the problem of scheduling classes in a classroom
- Many courses are candidates to be scheduled in that room, but not all can have it (can't hold two courses at once)
- Want to maximize utilization of the room in terms of number of classes scheduled
- This is an example of the **activity selection problem**:
  - Given: Set $S = \{a_1, a_2, \ldots, a_n\}$ of n proposed activities that wish to use a resource that can serve only one activity at a time
  - a_i has a **start time** s_i and a **finish time** f_i, $0 \le s_i < f_i < \infty$
  - If a_i is scheduled to use the resource, it occupies it during the interval $[s_i, f_i) \Rightarrow$ can schedule both a_i and a_j iff $s_i \ge f_j$ or $s_j \ge f_i$ (if this happens, then we say that a_i and a_j are **compatible**)
  - Goal is to find a largest subset $S' \subseteq S$ such that all activities in S' are pairwise compatible
  - Assume that activities are sorted by finish time:

$$f_1 \le f_2 \le \cdots \le f_n$$

Activity Selection (2)

i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

Sets of mutually compatible activities: $\{a_3, a_9, a_{11}\}$, $\{a_1, a_4, a_8, a_{11}\}$, $\{a_2, a_4, a_9, a_{11}\}$

Optimal Substructure of Activity Selection

- Let S_{ij} be set of activities that start after a_i finishes and that finish before a_j starts
- Let $A_{ij} \subseteq S_{ij}$ be a largest set of activities that are mutually compatible
- If activity $a_k \in A_{ij}$, then we get two subproblems: S_{ik} (subset starting after a_i finishes and finishing before a_k starts) and S_{kj}
- If we extract from A_{ij} its set of activities from S_{ik}, we get $A_{ik} = A_{ij} \cap S_{ik}$, which is an optimal solution to S_{ik}
  - If it weren't, then we could take the better solution to S_{ik} (call it A'_{ik}) and plug its tasks into A_{ij} and get a better solution
  - Works because subproblem S_{ik} independent from S_{kj}
- Thus if we pick an activity a_k to be in an optimal solution and then solve the subproblems, our optimal solution is $A_{ij} = A_{ik} \cup \{a_k\} \cup A_{kj}$, which is of size $|A_{ik}| + |A_{kj}| + 1$

Optimal Substructure Example

i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

- Let[1] $S_{ij} = S_{1,11} = \{a_1, \ldots, a_{11}\}$ and $A_{ij} = A_{1,11} = \{a_1, a_4, a_8, a_{11}\}$
- For $a_k = a_8$, get $S_{1k} = S_{1,8} = \{a_1, a_2, a_3, a_4\}$ and $S_{8,11} = \{a_{11}\}$
- $A_{1,8} = A_{1,11} \bigcap S_{1,8} = \{a_1, a_4\}$, which is optimal for $S_{1,8}$
- $A_{8,11} = A_{1,11} \bigcap S_{8,11} = \{a_{11}\}$, which is optimal for $S_{8,11}$

[1] Left-hand boundary condition addressed by adding to S activity a_0 with $f_0 = 0$ and setting $i = 0$

Recursive Definition

- Let $c[i,j]$ be the size of an optimal solution to S_{ij}

$$c[i,j] = \begin{cases} 0 & \text{if } S_{ij} = \emptyset \\ \max_{a_k \in S_{ij}} \{c[i,k] + c[k,j] + 1\} & \text{if } S_{ij} \neq \emptyset \end{cases}$$

- In dynamic programming, we need to try all a_k since we don't know which one is the best choice...
- ...or do we?

Greedy Choice (1)

- What if, instead of trying all activities a_k, we simply chose the one with the earliest finish time of all those still compatible with the scheduled ones?
- This is a **greedy choice** in that it maximizes the amount of time left over to schedule other activities
- Let $S_k = \{a_i \in S : s_i \geq f_k\}$ be set of activities that start after a_k finishes
- If we greedily choose a_1 first (with earliest finish time), then S_1 is the only subproblem to solve

Greedy Choice (2)

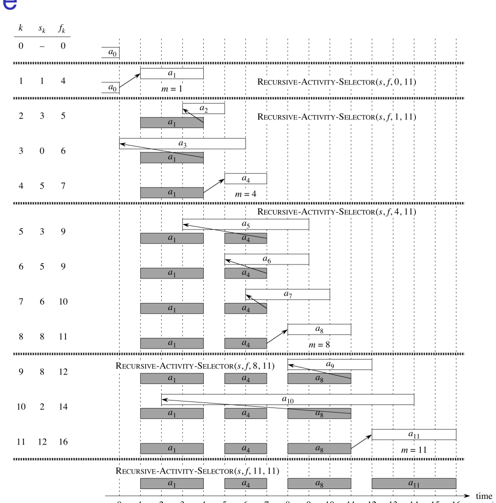
- **Theorem:** Consider any nonempty subproblem S_k and let a_m be an activity in S_k with earliest finish time. Then a_m is in **some** maximum-size subset of mutually compatible activities of S_k
- **Proof (by construction):**
  - Let A_k be an optimal solution to S_k and let a_j have earliest finish time of all in A_k
  - If $a_j = a_m$, we're done
  - If $a_j \neq a_m$, then define $A'_k = A_k \setminus \{a_j\} \cup \{a_m\}$
  - Activities in A' are mutually compatible since those in A are mutually compatible and $f_m \leq f_j$
  - Since $|A'_k| = |A_k|$, we get that A'_k is a maximum-size subset of mutually compatible activities of S_k that includes a_m □
- What this means is that **there exists** an optimal solution that uses the greedy choice

Greedy-Activity-Selector(s, f, n)

```
A = {a_1} ;
k = 1 ;
for m = 2 to n do
    if s[m] ≥ f[k] then
        A = A ∪ {a_m} ;
        k = m
end
return A
```

What is the time complexity?

Example



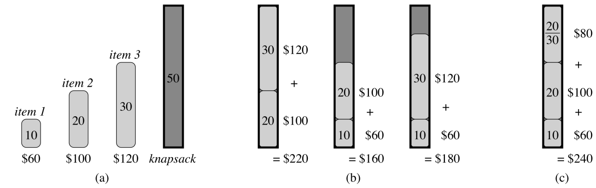
Greedy vs Dynamic Programming (1)

- Like with dynamic programming, greedy leverages a problem's **optimal substructure property**
- When can we get away with a greedy algorithm instead of DP?
- When we can argue that the **greedy choice** is part of an optimal solution, implying that we need not explore all subproblems
- Example: The **knapsack problem**
  - There are n items that a thief can steal, item i weighing w_i pounds and worth v_i dollars
  - The thief's goal is to steal a set of items weighing at most W pounds and maximizes total value
  - In the **0-1 knapsack problem**, each item must be taken in its entirety (e.g., gold bars)
  - In the **fractional knapsack problem**, the thief can take part of an item and get a proportional amount of its value (e.g., gold dust)

Greedy vs Dynamic Programming (2)

- There's a greedy algorithm for the fractional knapsack problem
  - Sort the items by v_i/w_i and choose the items in descending order
  - Has greedy choice property, since any optimal solution lacking the greedy choice can have the greedy choice swapped in
    - Works because one can always completely fill the knapsack at the last step
- Greedy strategy does not work for 0-1 knapsack, but do have $O(nW)$-time dynamic programming algorithm
  - Note that time complexity is *pseudopolynomial*
  - Decision problem is NP-complete

Greedy vs Dynamic Programming (3)



Problem instance 0-1 (greedy is suboptimal) Fractional

Huffman Coding

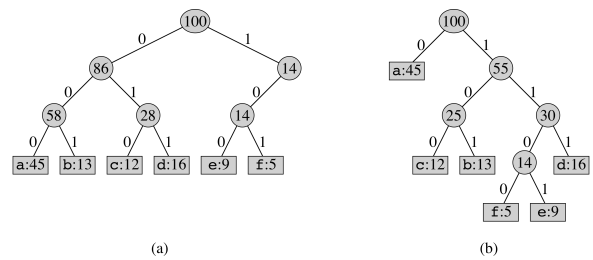
- Interested in encoding a file of symbols from some alphabet
- Want to minimize the size of the file, based on the frequencies of the symbols
- **Fixed-length code** uses $\lceil \log_2 n \rceil$ bits per symbol, where n is the size of the alphabet C
- **Variable-length code** uses fewer bits for more frequent symbols

	a	b	c	d	e	f
Frequency (in thousands)	45	13	12	16	9	5
Fixed-length codeword	000	001	010	011	100	101
Variable-length codeword	0	101	100	111	1101	1100

Fixed-length code uses 300k bits, variable-length uses 224k

Huffman Coding (2)

Can represent any encoding as a binary tree



If $c.freq$ = frequency of codeword and $d_T(c)$ = depth, cost of tree T is

$$B(T) = \sum_{c \in C} c.freq \cdot d_T(c)$$

Algorithm for Optimal Codes

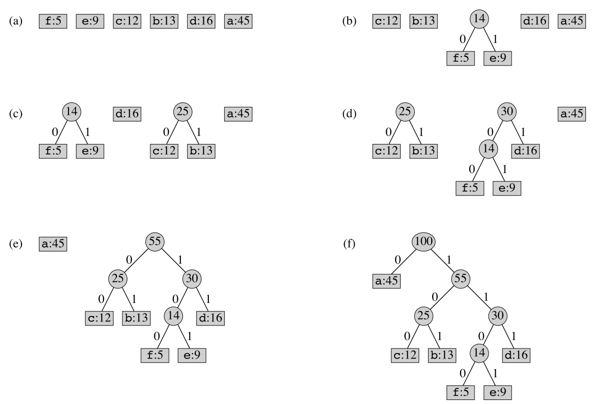
- Can get an optimal code by finding an appropriate **prefix code**, where no codeword is a prefix of another
- Optimal code also corresponds to a full binary tree
- Huffman's algorithm builds an optimal code by greedily building its tree
- Given alphabet C (which corresponds to leaves), find the two least frequent ones, merge them into a subtree
- Frequency of new subtree is the sum of the frequencies of its children
- Then add the subtree back into the set for future consideration

Huffman(C)

```
n = |C| ;
Q = C            // min-priority queue ;
for i = 1 to n − 1 do
    allocate node z ;
    z.left = x = EXTRACT-MIN(Q) ;
    z.right = y = EXTRACT-MIN(Q) ;
    z.freq = x.freq + y.freq ;
    INSERT(Q, z) ;
end
return EXTRACT-MIN(Q)     // return root ;
```

Time complexity: $n-1$ iterations, $O(\log n)$ time per iteration, total $O(n \log n)$

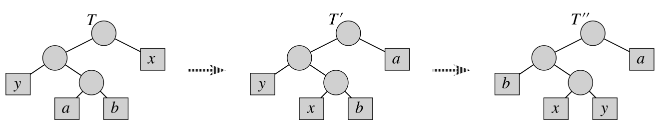
Huffman Example



Optimal Coding Has Greedy Choice Property (1)

- **Lemma:** Let C be an alphabet in which symbol $c \in C$ has frequency $c.freq$ and let $x, y \in C$ have lowest frequencies. Then there exists an optimal prefix code for C in which codewords for x and y have the same length and differ only in the last bit.
  - I.e., an optimal solution exists that merges lowest frequencies first
- **Proof:** Let T be a tree representing an arbitrary optimal prefix code, and let a and b be siblings of maximum depth in T
  - Assume, w.l.o.g., that $x.freq \le y.freq$ and $a.freq \le b.freq$
  - Since x and y are the two least frequent nodes, we get $x.freq \le a.freq$ and $y.freq \le b.freq$
  - Convert T to T' by exchanging a and x, then convert to T'' by exchanging b and y
  - In T'', x and y are siblings of maximum depth

Optimal Coding Has Greedy Choice Property (2)



Is T'' optimal?

Optimal Coding Has Greedy Choice Property (3)

Cost difference between T and T' is $B(T) - B(T')$:

$$\begin{aligned}
&= \sum_{c \in C} c.freq \cdot d_T(c) - \sum_{c \in C} c.freq \cdot d_{T'}(c) \\
&= x.freq \cdot d_T(x) + a.freq \cdot d_T(a) - x.freq \cdot d_{T'}(x) - a.freq \cdot d_{T'}(a) \\
&= x.freq \cdot d_T(x) + a.freq \cdot d_T(a) - x.freq \cdot d_T(a) - a.freq \cdot d_T(x) \\
&= (a.freq - x.freq)(d_T(a) - d_T(x)) \ge 0
\end{aligned}$$

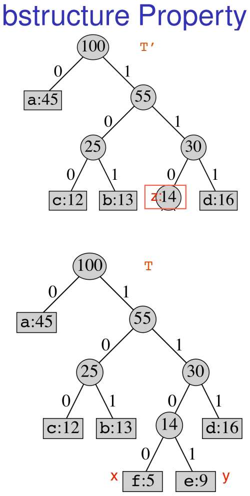
since $a.freq \ge x.freq$ and $d_T(a) \ge d_T(x)$
Similarly, $B(T') - B(T'') \ge 0$, so $B(T'') \le B(T)$, so T'' is optimal □

Optimal Coding Has Optimal Substructure Property (1)

Lemma:

- Let C be an alphabet in which symbol $c \in C$ has frequency $c.freq$ and let $x, y \in C$ have lowest frequencies
- Let $C' = C \setminus \{x, y\} \cup \{z\}$ and $z.freq = x.freq + y.freq$
- Let T' be any tree representing an optimal prefix code for C'

⇒ Then T, which is T' with leaf z replaced by internal node with children x and y, represents an optimal prefix code for C



Optimal Coding Has Optimal Substructure Property (2)

Proof:

- Since $d_T(x) = d_T(y) = d_{T'}(z) + 1$,

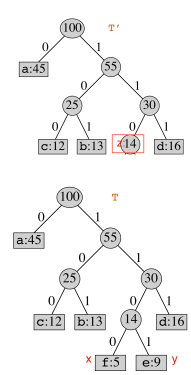
$$\begin{aligned}
& x.freq \cdot d_T(x) + y.freq \cdot d_T(y) \\
&= (x.freq + y.freq)(d_{T'}(z) + 1) \\
&= z.freq \cdot d_{T'}(z) + (x.freq + y.freq)
\end{aligned}$$

- Also, since $d_T(c) = d_{T'}(c)$ for all $c \in C \setminus \{x, y\}$,

$B(T) = B(T') + x.freq + y.freq$

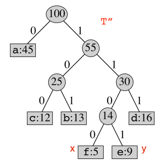
and

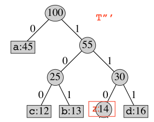
$B(T') = B(T) - x.freq - y.freq$



Optimal Coding Has Optimal Substructure Property (3)

- Assume that T is not optimal, i.e., $B(T'') < B(T)$ for some T''
- Assume w.l.o.g. (based on greedy choice lemma) that x and y are siblings in T''
- In T'', replace x, y, and parent with z such that $z.freq = x.freq + y.freq$, to get T''':





$$
\begin{aligned}
B(T''') &= B(T'') - x.freq - y.freq && \text{(prev. slide)} \\
&< B(T) - x.freq - y.freq && \text{(subopt assump)} \\
&= B(T') && \text{(prev. slide)}
\end{aligned}
$$

Contradicts assumption that T' is optimal for C' □