Computer Science & Engineering 423/823 Design and Analysis of Algorithms

Lecture 04 — Greedy Algorithms (Chapter 16)

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Introduction

- Greedy methods: A technique for solving **optimization problems**
  - Choose a solution to a problem that is best per an objective function
- Similar to dynamic programming in that we examine subproblems, exploiting optimal substructure property
- Key difference: In dynamic programming we considered **all** possible subproblems
- In contrast, a greedy algorithm at each step commits to just one subproblem, which results in its **greedy choice** (locally optimal choice)
- Examples: Minimum spanning tree, single-source shortest paths

Activity Selection (1)

- Consider the problem of scheduling classes in a classroom
- Many courses are candidates to be scheduled in that room, but not all can have it (can't hold two courses at once)
- Want to maximize utilization of the room in terms of number of classes scheduled
- This is an example of the **activity selection problem**:
  - Given: Set $S = \{a_1, a_2, \ldots, a_n\}$ of n proposed activities that wish to use a resource that can serve only one activity at a time
  - a_i has a **start time** s_i and a **finish time** f_i, $0 \le s_i < f_i < \infty$
  - If a_i is scheduled to use the resource, it occupies it during the interval $[s_i, f_i) \Rightarrow$ can schedule both a_i and a_j iff $s_i \ge f_j$ or $s_j \ge f_i$ (if this happens, then we say that a_i and a_j are **compatible**)
  - Goal is to find a largest subset $S' \subseteq S$ such that all activities in S' are pairwise compatible
  - Assume that activities are sorted by finish time:

$$f_1 \le f_2 \le \cdots \le f_n$$

Notes and Questions

Activity Selection (2)

i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

Sets of mutually compatible activities: $\{a_3, a_9, a_{11}\}$, $\{a_1, a_4, a_8, a_{11}\}$, $\{a_2, a_4, a_9, a_{11}\}$

Notes and Questions

Optimal Substructure of Activity Selection

- Let S_{ij} be set of activities that start after a_i finishes and that finish before a_j starts
- Let $A_{ij} \subseteq S_{ij}$ be a largest set of activities that are mutually compatible
- If activity $a_k \in A_{ij}$, then we get two subproblems: S_{ik} (subset starting after a_i finishes and finishing before a_k starts) and S_{kj}
- If we extract from A_{ij} its set of activities from S_{ik}, we get $A_{ik} = A_{ij} \cap S_{ik}$, which is an optimal solution to S_{ik}
  - If it weren't, then we could take the better solution to S_{ik} (call it A'_{ik}) and plug its tasks into A_{ij} and get a better solution
  - Works because subproblem S_{ik} independent from S_{kj}
- Thus if we pick an activity a_k to be in an optimal solution and then solve the subproblems, our optimal solution is $A_{ij} = A_{ik} \cup \{a_k\} \cup A_{kj}$, which is of size $|A_{ik}| + |A_{kj}| + 1$

Notes and Questions

Optimal Substructure Example

i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

- Let[1] $S_{ij} = S_{1,11} = \{a_1, \ldots, a_{11}\}$ and $A_{ij} = A_{1,11} = \{a_1, a_4, a_8, a_{11}\}$
- For $a_k = a_8$, get $S_{1k} = S_{1,8} = \{a_1, a_2, a_3, a_4\}$ and $S_{8,11} = \{a_{11}\}$
- $A_{1,8} = A_{1,11} \bigcap S_{1,8} = \{a_1, a_4\}$, which is optimal for $S_{1,8}$
- $A_{8,11} = A_{1,11} \bigcap S_{8,11} = \{a_{11}\}$, which is optimal for $S_{8,11}$

[1] Left-hand boundary condition addressed by adding to S activity a_0 with $f_0 = 0$ and setting $i = 0$

Notes and Questions

Recursive Definition

- Let $c[i, j]$ be the size of an optimal solution to S_{ij}

$$c[i,j] = \begin{cases} 0 & \text{if } S_{ij} = \emptyset \\ \max_{a_k \in S_{ij}} \{c[i,k] + c[k,j] + 1\} & \text{if } S_{ij} \neq \emptyset \end{cases}$$

- In dynamic programming, we need to try all a_k since we don't know which one is the best choice...
- ...or do we?

Notes and Questions

Greedy Choice (1)

- What if, instead of trying all activities a_k, we simply chose the one with the earliest finish time of all those still compatible with the scheduled ones?
- This is a **greedy choice** in that it maximizes the amount of time left over to schedule other activities
- Let $S_k = \{a_i \in S : s_i \geq f_k\}$ be set of activities that start after a_k finishes
- If we greedily choose a_1 first (with earliest finish time), then S_1 is the only subproblem to solve

Notes and Questions

Greedy Choice (2)

- **Theorem:** Consider any nonempty subproblem S_k and let a_m be an activity in S_k with earliest finish time. Then a_m is in **some** maximum-size subset of mutually compatible activities of S_k
- **Proof (by construction):**
  - Let A_k be an optimal solution to S_k and let a_j have earliest finish time of all in A_k
  - If $a_j = a_m$, we're done
  - If $a_j \neq a_m$, then define $A'_k = A_k \setminus \{a_j\} \cup \{a_m\}$
  - Activities in A' are mutually compatible since those in A are mutually compatible and $f_m \leq f_j$
  - Since $|A'_k| = |A_k|$, we get that A'_k is a maximum-size subset of mutually compatible activities of S_k that includes a_m □
- What this means is that **there exists** an optimal solution that uses the greedy choice

Notes and Questions

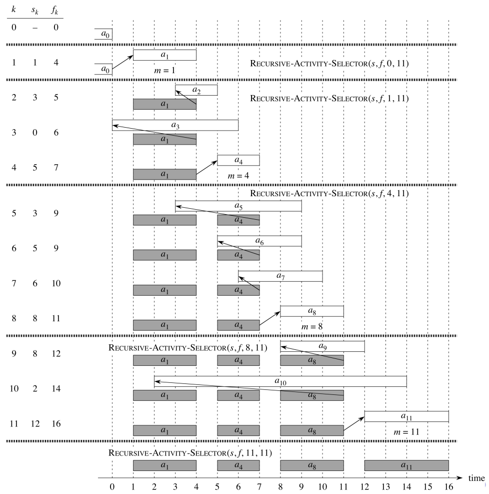
Greedy-Activity-Selector(s, f, n)

```
A = {a_1} ;
k = 1 ;
for m = 2 to n do
    if s[m] ≥ f[k] then
        A = A ∪ {a_m} ;
        k = m
end
return A
```

What is the time complexity?

Notes and Questions

Example



Notes and Questions

Greedy vs Dynamic Programming (1)

- Like with dynamic programming, greedy leverages a problem's **optimal substructure property**
- When can we get away with a greedy algorithm instead of DP?
- When we can argue that the **greedy choice** is part of an optimal solution, implying that we need not explore all subproblems
- Example: The **knapsack problem**
  - There are n items that a thief can steal, item i weighing w_i pounds and worth v_i dollars
  - The thief's goal is to steal a set of items weighing at most W pounds and maximizes total value
  - In the **0-1 knapsack problem**, each item must be taken in its entirety (e.g., gold bars)
  - In the **fractional knapsack problem**, the thief can take part of an item and get a proportional amount of its value (e.g., gold dust)

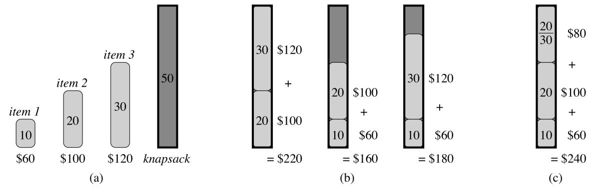
Notes and Questions

Greedy vs Dynamic Programming (2)

- There's a greedy algorithm for the fractional knapsack problem
  - Sort the items by v_i/w_i and choose the items in descending order
  - Has greedy choice property, since any optimal solution lacking the greedy choice can have the greedy choice swapped in
    - Works because one can always completely fill the knapsack at the last step
- Greedy strategy does not work for 0-1 knapsack, but do have $O(nW)$-time dynamic programming algorithm
  - Note that time complexity is *pseudopolynomial*
  - Decision problem is NP-complete

Notes and Questions

Greedy vs Dynamic Programming (3)



Problem instance | 0-1 (greedy is suboptimal) | Fractional

Notes and Questions

Huffman Coding

- Interested in encoding a file of symbols from some alphabet
- Want to minimize the size of the file, based on the frequencies of the symbols
- **Fixed-length code** uses $\lceil \log_2 n \rceil$ bits per symbol, where n is the size of the alphabet C
- **Variable-length code** uses fewer bits for more frequent symbols

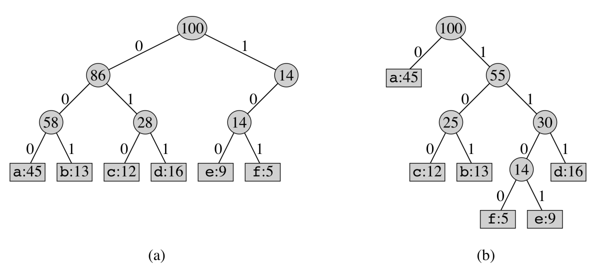
	a	b	c	d	e	f
Frequency (in thousands)	45	13	12	16	9	5
Fixed-length codeword	000	001	010	011	100	101
Variable-length codeword	0	101	100	111	1101	1100

Fixed-length code uses 300k bits, variable-length uses 224k

Notes and Questions

Huffman Coding (2)

Can represent any encoding as a binary tree



If $c.freq$ = frequency of codeword and $d_T(c)$ = depth, cost of tree T is

$$B(T) = \sum_{c \in C} c.freq \cdot d_T(c)$$

Notes and Questions

Algorithm for Optimal Codes

- Can get an optimal code by finding an appropriate **prefix code**, where no codeword is a prefix of another
- Optimal code also corresponds to a full binary tree
- Huffman's algorithm builds an optimal code by greedily building its tree
- Given alphabet C (which corresponds to leaves), find the two least frequent ones, merge them into a subtree
- Frequency of new subtree is the sum of the frequencies of its children
- Then add the subtree back into the set for future consideration

Notes and Questions

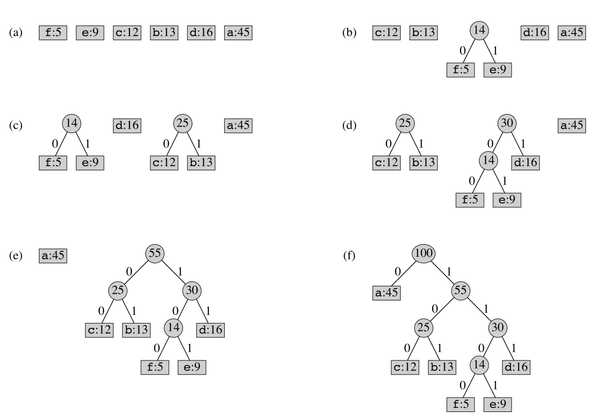
Huffman(C)

$n = |C|$;
$Q = C$ // min-priority queue ;
for $i = 1$ *to* $n - 1$ **do**
  allocate node z ;
  $z.left = x = \text{Extract-Min}(Q)$;
  $z.right = y = \text{Extract-Min}(Q)$;
  $z.freq = x.freq + y.freq$;
  $\text{Insert}(Q, z)$;
end
return $\text{Extract-Min}(Q)$ // return root ;

Time complexity: $n - 1$ iterations, $O(\log n)$ time per iteration, total $O(n \log n)$

Notes and Questions

Huffman Example



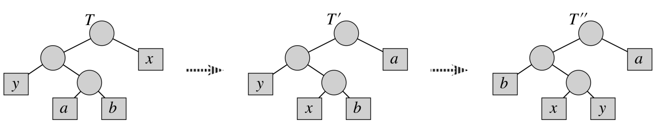
Notes and Questions

Optimal Coding Has Greedy Choice Property (1)

- **Lemma:** Let C be an alphabet in which symbol $c \in C$ has frequency $c.freq$ and let $x, y \in C$ have lowest frequencies. Then there exists an optimal prefix code for C in which codewords for x and y have the same length and differ only in the last bit.
  - I.e., an optimal solution exists that merges lowest frequencies first
- **Proof:** Let T be a tree representing an arbitrary optimal prefix code, and let a and b be siblings of maximum depth in T
  - Assume, w.l.o.g., that $x.freq \leq y.freq$ and $a.freq \leq b.freq$
  - Since x and y are the two least frequent nodes, we get $x.freq \leq a.freq$ and $y.freq \leq b.freq$
  - Convert T to T' by exchanging a and x, then convert to T'' by exchanging b and y
  - In T'', x and y are siblings of maximum depth

Notes and Questions

Optimal Coding Has Greedy Choice Property (2)



Is T'' optimal?

Notes and Questions

Optimal Coding Has Greedy Choice Property (3)

Cost difference between T and T' is $B(T) - B(T')$:

$$
\begin{aligned}
&= \sum_{c \in C} c.freq \cdot d_T(c) - \sum_{c \in C} c.freq \cdot d_{T'}(c) \\
&= x.freq \cdot d_T(x) + a.freq \cdot d_T(a) - x.freq \cdot d_{T'}(x) - a.freq \cdot d_{T'}(a) \\
&= x.freq \cdot d_T(x) + a.freq \cdot d_T(a) - x.freq \cdot d_T(a) - a.freq \cdot d_T(x) \\
&= (a.freq - x.freq)(d_T(a) - d_T(x)) \geq 0
\end{aligned}
$$

since $a.freq \geq x.freq$ and $d_T(a) \geq d_T(x)$
Similarly, $B(T') - B(T'') \geq 0$, so $B(T'') \leq B(T)$, so T'' is optimal □

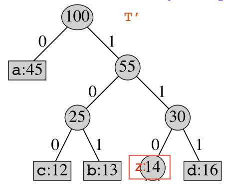
Notes and Questions

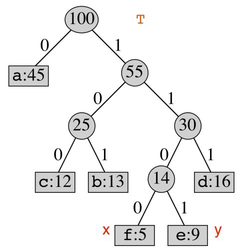
Optimal Coding Has Optimal Substructure Property (1)

Lemma:

- Let C be an alphabet in which symbol $c \in C$ has frequency $c.freq$ and let $x, y \in C$ have lowest frequencies
- Let $C' = C \setminus \{x, y\} \cup \{z\}$ and $z.freq = x.freq + y.freq$
- Let T' be any tree representing an optimal prefix code for C'

$\Rightarrow$ Then T, which is T' with leaf z replaced by internal node with children x and y, represents an optimal prefix code for C





Notes and Questions

Optimal Coding Has Optimal Substructure Property (2)

Proof:

- Since $d_T(x) = d_T(y) = d_{T'}(z) + 1$,

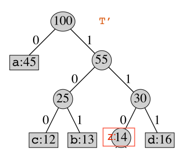
$$\begin{aligned} x.freq \cdot d_T(x) + y.freq \cdot d_T(y) \\ = (x.freq + y.freq)(d_{T'}(z) + 1) \\ = z.freq \cdot d_{T'}(z) + (x.freq + y.freq) \end{aligned}$$

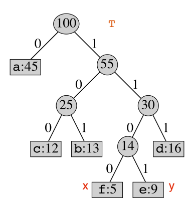
- Also, since $d_T(c) = d_{T'}(c)$ for all $c \in C \setminus \{x, y\}$,

  $B(T) = B(T') + x.freq + y.freq$

  and

  $B(T') = B(T) - x.freq - y.freq$

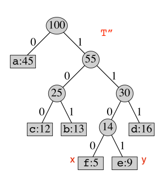


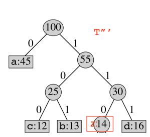


Notes and Questions

Optimal Coding Has Optimal Substructure Property (3)

- Assume that T is not optimal, i.e., $B(T'') < B(T)$ for some T''
- Assume w.l.o.g. (based on greedy choice lemma) that x and y are siblings in T''
- In T'', replace x, y, and parent with z such that $z.freq = x.freq + y.freq$, to get T''':





$$\begin{aligned} B(T''') &= B(T'') - x.freq - y.freq && \text{(prev. slide)} \\ &< B(T) - x.freq - y.freq && \text{(subopt assump)} \\ &= B(T') && \text{(prev. slide)} \end{aligned}$$

Contradicts assumption that T' is optimal for C' □

Notes and Questions