Computer Science & Engineering 423/823 Design and Analysis of Algorithms Lecture 07 — All-Pairs Shortest Paths (Chapter 25)

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Introduction

- Similar to SSSP, but find shortest paths for all pairs of vertices
- Given a weighted, directed graph G = (V, E) with weight function w : E → ℝ, find δ(u, v) for all (u, v) ∈ V × V
- One solution: Run an algorithm for SSSP |V| times, treating each vertex in V as a source
 - If no negative weight edges, use Dijkstra's algorithm, for time complexity of O(|V|³ + |V||E|) = O(|V|³) for array implementation, O(|V||E| log |V|) if heap used
 - ► If negative weight edges, use Bellman-Ford and get O(|V|²|E|) time algorithm, which is O(|V|⁴) if graph dense
- Can we do better?
 - Matrix multiplication-style algorithm: $\Theta(|V|^3 \log |V|)$
 - Floyd-Warshall algorithm: $\Theta(|V|^3)$
 - Both algorithms handle negative weight edges

Adjacency Matrix Representation

- Will use adjacency matrix representation
- Assume vertices are numbered: $V = \{1, 2, ..., n\}$
- Input to our algorithms will be $n \times n$ matrix W:

$$w_{ij} = \begin{cases} 0 & \text{if } i = j \\ \text{weight of edge } (i,j) & \text{if } (i,j) \in E \\ \infty & \text{if } (i,j) \notin E \end{cases}$$

- For now, assume negative weight cycles are absent
- In addition to distance matrices L and D produced by algorithms, can also build predecessor matrix Π, where π_{ij} = predecessor of j on a shortest path from i to j, or NIL if i = j or no path exists
 - Well-defined due to optimal substructure property

Print-All-Pairs-Shortest-Path (Π, i, j)

```
1 if i == j then

2 | print i

3 else if \pi_{ij} == NIL then

4 | print "no path from " i " to " j " exists"

5 else

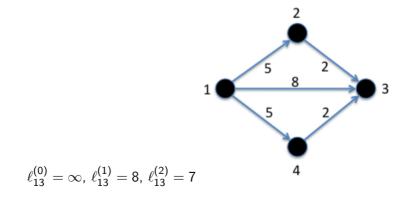
6 | PRINT-ALL-PAIRS-SHORTEST-PATH(\Pi, i, \pi_{ij})

7 | print j

8
```

Shortest Paths and Matrix Multiplication

- ▶ Will maintain a series of matrices $L^{(m)} = (\ell_{ij}^{(m)})$, where $\ell_{ij}^{(m)} =$ the minimum weight of any path from *i* to *j* that uses at most *m* edges
 - Special case: $\ell_{ij}^{(0)} = 0$ if i = j, ∞ otherwise



Recursive Solution

- Exploit optimal substructure property to get a recursive definition of $\ell_{ii}^{(m)}$
- ▶ To follow shortest path from *i* to *j* using at most *m* edges, either:
 - 1. Take shortest path from i to j using $\leq m-1$ edges and stay put, or
 - 2. Take shortest path from *i* to some *k* using $\leq m 1$ edges and traverse edge (k, j)

$$\ell_{ij}^{(m)} = \min\left(\ell_{ij}^{(m-1)}, \min_{1 \leq k \leq n}\left(\ell_{ik}^{(m-1)} + w_{kj}
ight)
ight)$$

• Since $w_{jj} = 0$ for all j, simplify to

$$\ell_{ij}^{(m)} = \min_{1 \le k \le n} \left(\ell_{ik}^{(m-1)} + w_{kj} \right)$$

► If no negative weight cycles, then since all shortest paths have ≤ n - 1 edges,

$$\delta(i,j) = \ell_{ij}^{(n-1)} = \ell_{ij}^{(n)} = \ell_{ij}^{(n+1)} = \cdots$$

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Bottum-Up Computation of L Matrices

- Start with weight matrix W and compute series of matrices L⁽¹⁾, L⁽²⁾,..., L⁽ⁿ⁻¹⁾
- Core of the algorithm is a routine to compute $L^{(m+1)}$ given $L^{(m)}$ and W
- Start with L⁽¹⁾ = W, and iteratively compute new L matrices until we get L⁽ⁿ⁻¹⁾

- Why is $L^{(1)} == W$?
- Can we detect negative-weight cycles with this algorithm? How?

Extend-Shortest-Paths(L, W)

1 n = number of rows of L // This is $L^{(m)}$ 2 create new $n \times n$ matrix L' // This will be $L^{(m+1)}$ 3 for i = 1 to n do for j = 1 to n do 4 $| \ell'_{ii} = \infty$ 5 for k = 1 to n do 6 $| \qquad | \qquad \ell'_{ii} = \min\left(\ell'_{ii}, \ell_{ik} + w_{ki}\right)$ 7 end 8 end 9 10 end 11 return L'

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Slow-All-Pairs-Shortest-Paths(W)

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1 n = number of rows of W

2 L^{(1)} = W

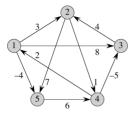
3 for m = 2 to n - 1 do

4 | L^{(m)} = \text{EXTEND-SHORTEST-PATHS}(L^{(m-1)}, W)

5 end

6 return L^{(n-1)}
```

Example



$$L^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad L^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{pmatrix}$$
$$L^{(3)} = \begin{pmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \qquad L^{(4)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

Improving Running Time

- ▶ What is time complexity of SLOW-ALL-PAIRS-SHORTEST-PATHS?
- Can we do better?
- Note that if, in EXTEND-SHORTEST-PATHS, we change + to multiplication and min to +, get matrix multiplication of L and W
- ► If we let ⊙ represent this "multiplication" operator, then SLOW-ALL-PAIRS-SHORTEST-PATHS computes

$$L^{(2)} = L^{(1)} \odot W = W^{(2)},$$

$$L^{(3)} = L^{(2)} \odot W = W^{(3)},$$

$$\vdots$$

$$L^{(n-1)} = L^{(n-2)} \odot W = W^{n-1}$$

► Thus, we get *L*^(*n*-1) by iteratively "multiplying" *W* via EXTEND-SHORTEST-PATHS

Improving Running Time (2)

- But we don't need every $L^{(m)}$; we only want $L^{(n-1)}$
- E.g., if we want to compute 7⁶⁴, we could multiply 7 by itself 64 times, or we could square it 6 times
- In our application, once we have a handle on L^{((n-1)/2)}, we can immediately get L⁽ⁿ⁻¹⁾ from one call to EXTEND-SHORTEST-PATHS(L^{((n-1)/2)}, L^{((n-1)/2)})
- Of course, we can similarly get $L^{((n-1)/2)}$ from "squaring" $L^{((n-1)/4)}$, and so on
- ▶ Starting from the beginning, we initialize $L^{(1)} = W$, then compute $L^{(2)} = L^{(1)} \odot L^{(1)}$, $L^{(4)} = L^{(2)} \odot L^{(2)}$, $L^{(8)} = L^{(4)} \odot L^{(4)}$, and so on
- What happens if n 1 is not a power of 2 and we "overshoot" it?
- How many steps of repeated squaring do we need to make?
- What is time complexity of this new algorithm?

Faster-All-Pairs-Shortest-Paths(W)

1 n = number of rows of W2 $L^{(1)} = W$ 3 m = 14 while m < n - 1 do 5 $\begin{vmatrix} L^{(2m)} = \text{EXTEND-SHORTEST-PATHS}(L^{(m)}, L^{(m)}) \\ m = 2m$ 7 end 8 return $L^{(m)}$

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Floyd-Warshall Algorithm

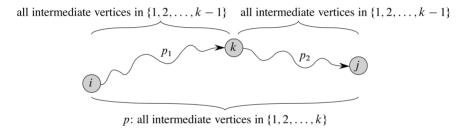
- Shaves the logarithmic factor off of the previous algorithm
- As with previous algorithm, start by assuming that there are no negative weight cycles; can detect negative weight cycles the same way as before
- Considers a different way to decompose shortest paths, based on the notion of an *intermediate vertex*
 - If simple path p = ⟨v₁, v₂, v₃,..., v_{ℓ-1}, v_ℓ⟩, then the set of intermediate vertices is {v₂, v₃,..., v_{ℓ-1}}

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Structure of Shortest Path

- Again, let $V = \{1, \ldots, n\}$, and fix $i, j \in V$
- ▶ For some $1 \le k \le n$, consider set of vertices $V_k = \{1, ..., k\}$
- Now consider all paths from i to j whose intermediate vertices come from V_k and let p be a minimum-weight path from them
- ► Is k ∈ p?
 - 1. If not, then all intermediate vertices of p are in V_{k-1} , and a SP from i to j based on V_{k-1} is also a SP from i to j based on V_k
 - 2. If so, then we can decompose p into $i \xrightarrow{p_1} k \xrightarrow{p_2} j$, where p_1 and p_2 are each shortest paths based on V_{k-1}

Structure of Shortest Path (2)



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Recursive Solution

- What does this mean?
- It means that a shortest path from i to j based on V_k is either going to be the same as that based on V_{k-1}, or it is going to go through k
- ► In the latter case, a shortest path from *i* to *j* based on V_k is going to be a shortest path from *i* to *k* based on V_{k-1}, followed by a shortest path from *k* to *j* based on V_{k-1}
- Let matrix $D^{(k)} = (d_{ij}^{(k)})$, where $d_{ij}^{(k)} =$ weight of a shortest path from *i* to *j* based on V_k :

$$d_{ij}^{(k)} = \left\{ egin{array}{ll} w_{ij} & ext{if } k = 0 \ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}
ight) & ext{if } k \geq 1 \end{array}
ight.$$

▶ Since all SPs are based on $V_n = V$, we get $d_{ij}^{(n)} = \delta(i,j)$ for all $i,j \in V$

Floyd-Warshall(W)

n = number of rows of W $D^{(0)} = W$ 3 for k = 1 to n do $\left| \begin{array}{c} \text{for } i = 1 \text{ to } n \text{ do} \\ 5 \\ 6 \\ 6 \\ 7 \\ 8 \end{array} \right| \left| \begin{array}{c} \text{for } j = 1 \text{ to } n \text{ do} \\ d_{ij}^{(k)} = \min \left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \right) \\ 0 \\ 8 \\ 8 \\ 1 \\ 10 \\ \text{return } D^{(n)} \end{array}$

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Transitive Closure

- Used to determine whether paths exist between pairs of vertices
- ▶ Given directed, unweighted graph G = (V, E) where V = {1,..., n}, the *transitive closure* of G is G* = (V, E*), where

 $E^* = \{(i, j) : \text{there is a path from } i \text{ to } j \text{ in } G\}$

- How can we directly apply Floyd-Warshall to find E^* ?
- Simpler way: Define matrix T similarly to D:

$$t_{ij}^{(0)} = \left\{egin{array}{ll} 0 & ext{if } i
eq j ext{ and } (i,j)
eq E \ 1 & ext{if } i = j ext{ or } (i,j) \in E \ t_{ij}^{(k)} = t_{ij}^{(k-1)} \lor \left(t_{ik}^{(k-1)} \land t_{kj}^{(k-1)}
ight)$$

► I.e., you can reach j from i using V_k if you can do so using V_{k-1} or if you can reach k from i and reach j from k, both using V_{k-1}

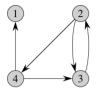
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Transitive-Closure(G)

1 allocate and initialize $n \times n$ matrix $T^{(0)}$ 2 for k = 1 to n do 3 | allocate $n \times n$ matrix $T^{(k)}$ 4 | for i = 1 to n do 5 | | for j = 1 to n do 6 | | $t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)}$ 7 | end 8 | end 9 end 10 return $T^{(n)}$

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Example



Analysis

- Like Floyd-Warshall, time complexity is officially $\Theta(n^3)$
- However, use of 0s and 1s exclusively allows implementations to use bitwise operations to speed things up significantly, processing bits in batch, a word at a time
- Also saves space
- Another space saver: Can update the T matrix (and F-W's D matrix) in place rather than allocating a new matrix for each step (Exercise 25.2-4)

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