Computer Science & Engineering 423/823
Design and Analysis of Algorithms

Lecture 05 — Minimum-Weight Spanning Trees (Chapter 23)

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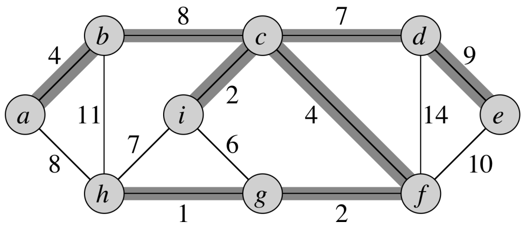
Introduction

- Given a connected, undirected graph $G = (V, E)$, a **spanning tree** is an acyclic subset $T \subseteq E$ that connects all vertices in V
  - T acyclic $\Rightarrow$ a tree
  - T connects all vertices $\Rightarrow$ **spans** G
- If G is weighted, then T's weight is $w(T) = \sum_{(u,v) \in T} w(u, v)$
- A **minimum weight spanning tree** (or **minimum spanning tree**, or MST) is a spanning tree of minimum weight
  - Not necessarily unique
- Applications: anything where one needs to connect all nodes with minimum cost, e.g., wires on a circuit board or fiber cable in a network

MST Example



Kruskal's Algorithm

- Greedy algorithm: Make the locally best choice at each step
- Starts by declaring each vertex to be its own tree (so all nodes together make a forest)
- Iteratively identify the minimum-weight edge (u, v) that connects two distinct trees, and add it to the MST T, merging u's tree with v's tree

MST-Kruskal(G, w)

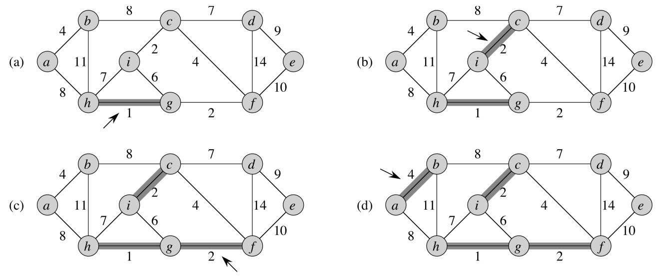
```
A = ∅
for each vertex v ∈ V do
    Make-Set(v)
end
sort edges in E into nondecreasing order by weight w
for each edge (u, v) ∈ E, taken in nondecreasing order
do
    if Find-Set(u) ≠ Find-Set(v) then
        A = A ∪ {(u, v)}
        Union(u, v)

end
return A
```

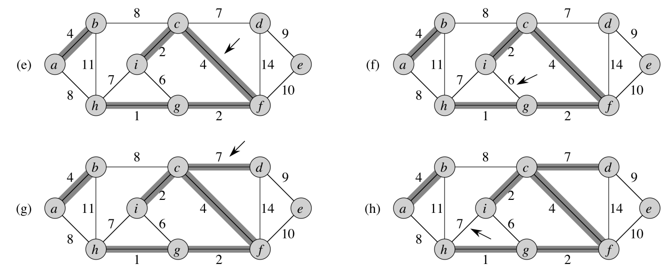
More on Kruskal's Algorithm

- Find-Set(u) returns a representative element from the set (tree) that contains u
- Union(u, v) combines u's tree to v's tree
- These functions are based on the **disjoint-set data structure**
- More on this later

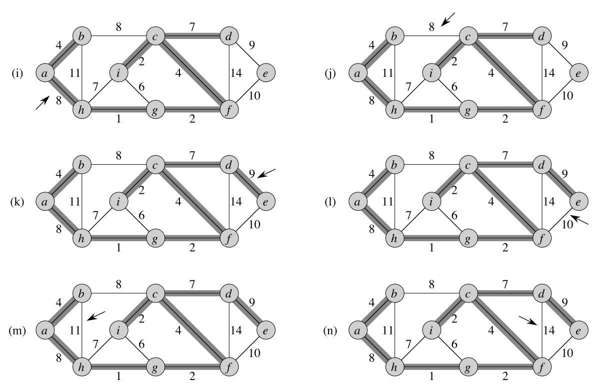
Example (1)



Example (2)



Example (3)



Disjoint-Set Data Structure

- Given a **universe** $U = \{x_1, \dots, x_n\}$ of elements (e.g., the vertices in a graph G), a DSDS maintains a collection $\mathcal{S} = \{S_1, \dots, S_k\}$ of disjoint sets of elements such that
  - Each element x_i is in exactly one set S_j
  - No set S_j is empty
- Membership in sets is dynamic (changes as program progresses)
- Each set $S \in \mathcal{S}$ has a **representative element** $x \in S$
- Chapter 21

Disjoint-Set Data Structure (2)

- DSDS implementations support the following functions:
  - $\textsc{Make-Set}(x)$ takes element x and creates new set $\{x\}$; returns pointer to x as set's representative
  - $\textsc{Union}(x, y)$ takes x's set (S_x) and y's set (S_y, assumed disjoint from S_x), merges them, destroys S_x and S_y, and returns representative for new set from $S_x \cup S_y$
  - $\textsc{Find-Set}(x)$ returns a pointer to the representative of the unique set that contains x
- Section 21.3: can perform d D-S operations on e elements in time $O(d\,\alpha(e))$, where $\alpha(e) = o(\lg^* e) = o(\log e)$ is *very* slowly growing:

$$\alpha(e) = \begin{cases} 0 & \text{if } 0 \le e \le 2 \\ 1 & \text{if } e = 3 \\ 2 & \text{if } 4 \le e \le 7 \\ 3 & \text{if } 8 \le e \le 2047 \\ 4 & \text{if } 2048 \le e \le 2^{2048} \ (\gg 10^{600}) \end{cases} \qquad \lg^*(e) = \begin{cases} 0 & \text{if } e \le 1 \\ 1 & \text{if } 1 < e \le 2 \\ 2 & \text{if } 2 < e \le 4 \\ 3 & \text{if } 4 < e \le 16 \\ 4 & \text{if } 16 < e \le 65536 \\ 5 & \text{if } 65536 < e \le 2^{65536} \end{cases}$$

Analysis of Kruskal's Algorithm

- Sorting edges takes time $O(|E| \log |E|)$
- Number of disjoint-set operations is $O(|V| + |E|)$ on $O(|V|)$ elements, which can be done in time $O((|V| + |E|)\,\alpha(|V|)) = O(|E|\,\alpha(|V|))$ since $|E| \ge |V| - 1$
- Since $\alpha(|V|) = o(\log |V|) = O(\log |E|)$, we get total time of $O(|E| \log |E|) = O(|E| \log |V|)$ since $\log |E| = O(\log |V|)$

Prim's Algorithm

- Greedy algorithm, like Kruskal's
- In contrast to Kruskal's, Prim's algorithm maintains a single tree rather than a forest
- Starts with an arbitrary tree root r
- Repeatedly finds a minimum-weight edge that is incident to a node not yet in tree

MST-Prim(G, w, r)

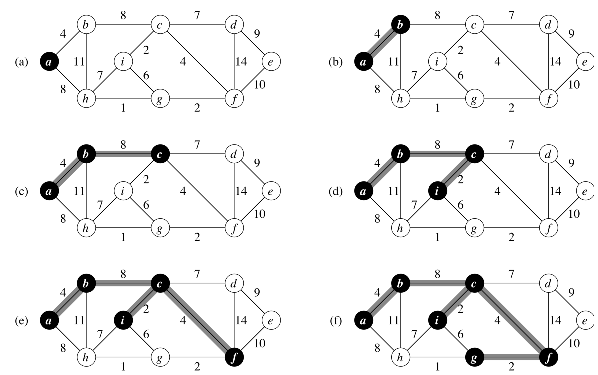
$A = \emptyset$
for *each vertex* $v \in V$ **do**
    $key[v] = \infty$
    $\pi[v] = \text{NIL}$
end
$key[r] = 0$
$Q = V$
while $Q \neq \emptyset$ **do**
    $u = \text{Extract-Min}(Q)$
    for *each* $v \in Adj[u]$ **do**
        if $v \in Q$ *and* $w(u, v) < key[v]$ **then**
            $\pi[v] = u$
            $key[v] = w(u, v)$

    end
end

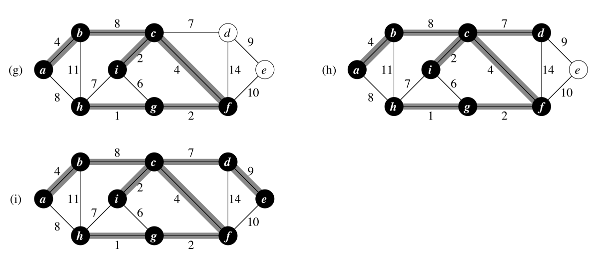
More on Prim's Algorithm

- $key[v]$ is the weight of the minimum weight edge from v to any node already in MST
- Extract-Min uses a **minimum heap** (minimum priority queue) data structure
  - Binary tree where the key at each node is $\leq$ keys of its children
  - Thus minimum value always at top
  - Any subtree is also a heap
  - Height of tree is $\Theta(\log n)$
  - Can build heap on n elements in $O(n)$ time
  - After returning the minimum, can filter new minimum to top in time $O(\log n)$
  - Based on Chapter 6

Example (1)



Example (2)

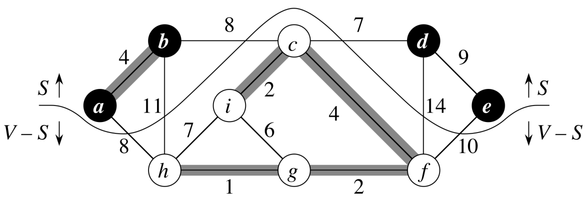


Analysis of Prim's Algorithm

- **Invariant:** Prior to each iteration of the while loop:
  1. Nodes already in MST are exactly those in $V \setminus Q$
  2. For all vertices $v \in Q$, if $\pi[v] \neq \text{NIL}$, then $key[v] < \infty$ and $key[v]$ is the weight of the lightest edge that connects v to a node already in the tree
- Time complexity:
  - Building heap takes time $O(|V|)$
  - Make $|V|$ calls to Extract-Min, each taking time $O(\log |V|)$
  - For loop iterates $O(|E|)$ times
    - In for loop, need constant time to check for queue membership and $O(\log |V|)$ time for decreasing v's key and updating heap
  - Yields total time of $O(|V| \log |V| + |E| \log |V|) = O(|E| \log |V|)$
  - Can decrease total time to $O(|E| + |V| \log |V|)$ using Fibonacci heaps

Proof of Correctness of Both Algorithms

- Both algorithms use greedy approach for optimality
- Maintain **invariant** that at any time, set of edges A selected so far is subset of some MST
  - $\Rightarrow$ Optimal substructure property
- Each iteration of each algorithm looks for a **safe edge** e such that $A \cup \{e\}$ is also a subset of an MST
  - $\Rightarrow$ Greedy choice
- Prove invariant via use of **cut** $(S, V - S)$ that **respects** A (no edges span cut)



Proof of Correctness of Both Algorithms (2)

- **Theorem:** Let $A \subseteq E$ be included in some MST of G, $(S, V - S)$ be a cut respecting A, and $(u, v) \in E$ be a minimum-weight edge crossing cut. Then (u, v) is a safe edge for A.
- **Proof:**
  - Let T be an MST including A and not including (u, v)
  - Let p be path from u to v in T, and (x, y) be edge from p crossing cut ($\Rightarrow$ not in A)
  - Since T is a spanning tree, so is $T' = T - \{(x, y)\} \cup \{(u, v)\}$
  - Both (u, v) and (x, y) cross cut, so $w(u, v) \leq w(x, y)$
  - So, $w(T') = w(T) - w(x, y) + w(u, v) \leq w(T)$
  - $\Rightarrow$ T' is MST
  - $\Rightarrow$ (u, v) safe for A since $A \cup \{(u, v)\} \subseteq T'$ $\square$

Proof of Correctness of Both Algorithms (3)