Computer Science & Engineering 423/823
Design and Analysis of Algorithms
Lecture 01 — Shall We Play A Game?

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Introduction

- In this course, I assume that you have learned several fundamental concepts on basic data structures and algorithms
- Let's confirm this
- What do I mean ...

… when I say: "Asymptotic Notation"

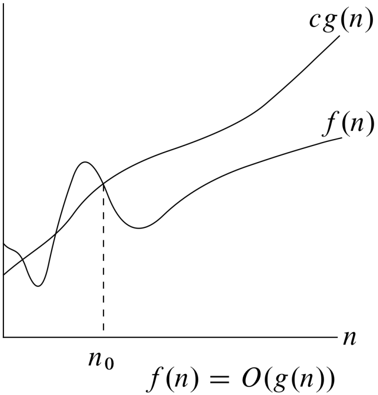
- A convenient means to succinctly express the growth of functions
  - Big-O
  - Big-Ω
  - Big-Θ
  - Little-o
  - Little-ω
- Important distinctions between these (**not interchangeable**)

Asymptotic Notation

... when I say: "Big-O"

Asymptotic upper bound

$$O(g(n)) = \{f(n) : \exists c, n_0 > 0 \text{ s.t. } \forall n \geq n_0, 0 \leq f(n) \leq c\,g(n)\}$$



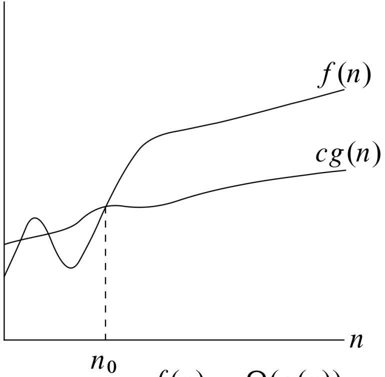
Can **very loosely and informally** think of this as a "$\leq$" relation between functions

Asymptotic Notation

... when I say: "Big-Ω"

Asymptotic lower bound

$$\Omega(g(n)) = \{f(n) : \exists c, n_0 > 0 \text{ s.t. } \forall n \geq n_0, 0 \leq c\, g(n) \leq f(n)\}$$



$f(n) = \Omega(g(n))$

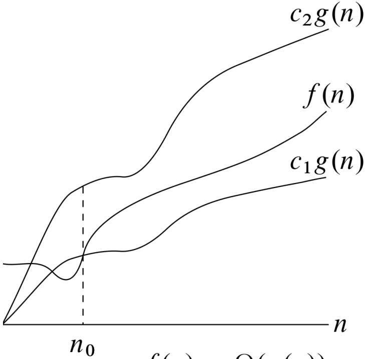
Can **very loosely and informally** think of this as a "$\geq$" relation between functions

Asymptotic Notation

... when I say: "Big-Θ"

Asymptotic tight bound

$$\Theta(g(n)) = \{f(n) : \exists c_1, c_2, n_0 > 0 \text{ s.t. } \forall n \geq n_0, 0 \leq c_1\, g(n) \leq f(n) \leq c_2\, g(n)\}$$



$f(n) = \Theta(g(n))$

Can **very loosely and informally** think of this as a "=" relation between functions

Asymptotic Notation

... when I say: "Little-o"

Upper bound, not asymptotically tight

$$o(g(n)) = \{f(n) : \forall c > 0, \exists n_0 > 0 \text{ s.t. } \forall n \geq n_0, 0 \leq f(n) < c\, g(n)\}$$

Upper inequality strict, and holds for **all** $c > 0$

Can **very loosely and informally** think of this as a "$<$" relation between functions

Asymptotic Notation

... when I say: "Little-ω"

Lower bound, not asymptotically tight

$$\omega(g(n)) = \{f(n) : \forall c > 0, \exists n_0 > 0 \text{ s.t. } \forall n \geq n_0, 0 \leq c\, g(n) < f(n)\}$$

$f(n) \in \omega(g(n)) \Leftrightarrow g(n) \in o(f(n))$
Can **very loosely and informally** think of this as a "$>$" relation between functions

... when I say: "Upper and Lower Bounds"

- Most often, we analyze algorithms and problems in terms of **time complexity** (number of operations)
- Sometimes we analyze in terms of **space complexity** (amount of memory)
- Can think of **upper** and **lower** bounds of time/space for a specific **algorithm** or a general **problem**

Upper and Lower Bounds

… when I say: “Upper Bound of an Algorithm”

- The most common form of analysis
- An algorithm A has an **upper bound** of $f(n)$ for input of size n if there exists **no input** of size n such that A requires more than $f(n)$ time
- E.g., we know from prior courses that Quicksort and Bubblesort take no more time than $O(n^2)$, while Mergesort has an upper bound of $O(n \log n)$
  - (But why is Quicksort used more in practice?)
- **Aside:** An algorithm's lower bound (not typically as interesting) is like a best-case result

Upper and Lower Bounds

… when I say: "Upper Bound of a Problem"

- A problem has an **upper bound** of $f(n)$ if there exists **at least one** algorithm that has an upper bound of $f(n)$
  - I.e., there exists an algorithm with time/space complexity of at most $f(n)$ on **all** inputs of size n
- E.g., since Mergesort has worst-case time complexity of $O(n \log n)$, the problem of sorting has an upper bound of $O(n \log n)$
  - Sorting also has an upper bound of $O(n^2)$ thanks to Bubblesort and Quicksort, but this is subsumed by the tighter bound of $O(n \log n)$

Upper and Lower Bounds

... when I say: "Lower Bound of a Problem"

- A problem has a **lower bound** of $f(n)$ if, for **any** algorithm A to solve the problem, there exists **at least one** input of size n that forces A to take at least $f(n)$ time/space
- This pathological input depends on the specific algorithm A
- E.g., there is an input of size n (reverse order) that forces Bubblesort to take $\Omega(n^2)$ steps
- Also e.g., there is a different input of size n that forces Mergesort to take $\Omega(n \log n)$ steps, but none exists forcing $\omega(n \log n)$ steps
- Since **every** sorting algorithm has an input of size n forcing $\Omega(n \log n)$ steps, the sorting problem has a **time complexity lower bound** of $\Omega(n \log n)$

  $\Rightarrow$ Mergesort is **asymptotically optimal**

Upper and Lower Bounds

… when I say: "Lower Bound of a Problem" (2)

- To argue a lower bound for a problem, can use an **adversarial** argument: An algorithm that simulates **arbitrary** algorithm A to build a pathological input
- Needs to be in some general (algorithmic) form since the nature of the pathological input depends on the specific algorithm A
- Can also **reduce** one problem to another to establish lower bounds
  - **Spoiler Alert:** This semester we will show that if we can compute convex hull in $o(n \log n)$ time, then we can also sort in time $o(n \log n)$; this cannot be true, so convex hull takes time $\Omega(n \log n)$

… when I say: "Efficiency"

- We say that an algorithm is **time-** or **space-efficient** if its worst-case time (space) complexity is $O(n^c)$ for constant c for input size n
- I.e., polynomial in the size of the input
- **Note on input size:** We measure the size of the input in terms of the **number of bits** needed to represent it
  - E.g., a graph of n nodes takes $O(n \log n)$ bits to represent the nodes and $O(n^2 \log n)$ bits to represent the edges
    - Thus, an algorithm that runs in time $O(n^c)$ is efficient
  - In contrast, a problem that includes as an input a numeric parameter k (e.g., threshold) only needs $O(\log k)$ bits to represent
    - In this case, an efficient algorithm for this problem **must** run in time $O(\log^c k)$
    - If instead polynomial in k, sometimes call this **pseudopolynomial**

… when I say: "Recurrence Relations"

- We know how to analyze non-recursive algorithms to get asymptotic bounds on run time, but what about recursive ones like Mergesort and Quicksort?
- We use a **recurrence relation** to capture the time complexity and then bound the relation asymptotically
- E.g., Mergesort splits the input array of size n into two sub-arrays, recursively sorts each, and then merges the two sorted lists into a single, sorted one
- If $T(n)$ is time for Mergesort on n elements,

$$T(n) = 2T(n/2) + O(n)$$

- Still need to get an asymptotic bound on $T(n)$

Recurrence Relations

... when I say: "Master Theorem" or "Master Method"

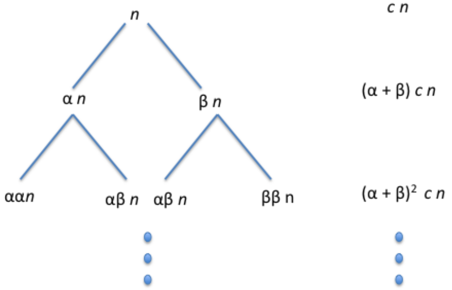
- **Theorem:** Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined as $T(n) = aT(n/b) + f(n)$. Then $T(n)$ is bounded as follows:
    1. If $f(n) = O(n^{\log_b a - \epsilon})$ for constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
    2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
    3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for constant $c < 1$ and sufficiently large n, then $T(n) = \Theta(f(n))$
- E.g., for Mergesort, can apply theorem with $a = b = 2$, use case 2, and get $T(n) = \Theta\left(n^{\log_2 2} \log n\right) = \Theta\left(n \log n\right)$

Recurrence Relations

Other Approaches

Theorem: For recurrences of the form $T(\alpha n) + T(\beta n) + O(n)$ for $\alpha + \beta < 1$, $T(n) = O(n)$

Proof: Top $T(n)$ takes $O(n)$ time ($= cn$ for some constant c). Then calls to $T(\alpha n)$ and $T(\beta n)$, which take a total of $(\alpha + \beta)cn$ time, and so on



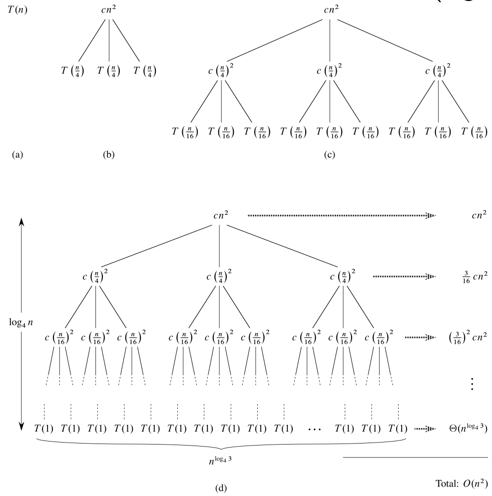
Summing these infinitely yields (since $\alpha + \beta < 1$)

$$cn(1 + (\alpha + \beta) + (\alpha + \beta)^2 + \cdots) = \frac{cn}{1 - (\alpha + \beta)} = c'n = O(n)$$

Recurrence Relations

Still Other Approaches

Previous theorem special case of **recursion-tree method**: (e.g., $T(n) = 3T(n/4) + O(n^2)$)

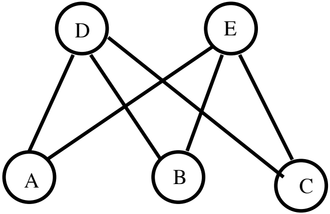


Another approach is **substitution method** (guess and prove via induction)

Graphs

… when I say: “(Undirected) Graph”

A **(simple, or undirected)** graph $G = (V, E)$ consists of V, a nonempty set of vertices and E a set of **unordered** pairs of distinct vertices called **edges**



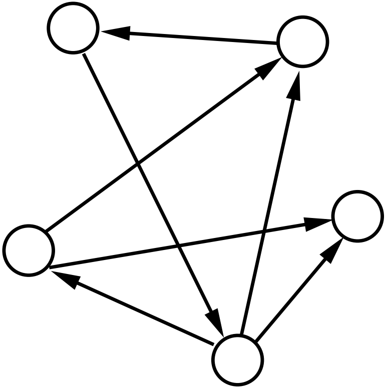
V={A,B,C,D,E}

E={ (A,D),(A,E),(B,D),
(B,E),(C,D),(C,E)}

Graphs

… when I say: "Directed Graph"

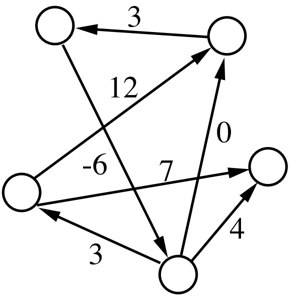
A **directed** graph (digraph) $G = (V, E)$ consists of V, a nonempty set of vertices and E a set of *ordered* pairs of distinct vertices called *edges*



Graphs

... when I say: "Weighted Graph"

A **weighted** graph is an undirected or directed graph with the additional property that each edge e has associated with it a real number $w(e)$ called its *weight*



Graphs

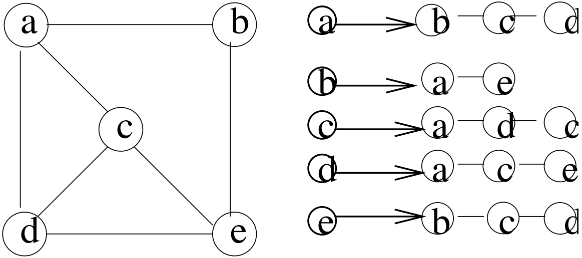
... when I say: "Representations of Graphs"

- Two common ways of representing a graph: **Adjacency list** and **adjacency matrix**
- Let $G = (V, E)$ be a graph with n vertices and m edges

Graphs

… when I say: "Adjacency List"

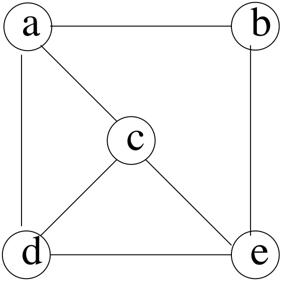
- For each vertex $v \in V$, store a list of vertices adjacent to v
- For weighted graphs, add information to each node
- How much is space required for storage?



Graphs

... when I say: "Adjacency Matrix"

- Use an $n \times n$ matrix M, where $M(i,j) = 1$ if (i,j) is an edge, 0 otherwise
- If G weighted, store weights in the matrix, using ∞ for non-edges
- How much is space required for storage?



$$\begin{array}{c|ccccc|}
 & a & b & c & d & e \\
\hline
a & 0 & 1 & 1 & 1 & 0 \\
b & 1 & 0 & 0 & 0 & 1 \\
c & 1 & 0 & 0 & 1 & 1 \\
d & 1 & 0 & 1 & 0 & 1 \\
e & 0 & 1 & 1 & 1 & 0 \\
\hline
\end{array}$$

Algorithmic Techniques

… when I say: "Dynamic Programming"

- **Dynamic programming** is a technique for solving **optimization problems**, where we need to choose a "best" solution, as evaluated by an **objective function**
- **Key element:** Decompose a problem into **subproblems**, optimally solve them recursively, and then combine the solutions into a final (optimal) solution
- **Important component:** There are typically an exponential number of subproblems to solve, but many of them overlap
  - $\Rightarrow$ Can re-use the solutions rather than re-solving them
- Number of distinct subproblems is polynomial
- Works for problems that have the **optimal substructure property**, in that an optimal solution is made up of optimal solutions to subproblems
  - Can find optimal solution if we consider all possible subproblems
- Example: All-pairs shortest paths

Algorithmic Techniques

… when I say: "Greedy Algorithms"

- Another optimization technique
- Similar to dynamic programming in that we examine subproblems, exploiting optimal substructure property
- **Key difference:** In dynamic programming we considered all possible subproblems
- In contrast, a greedy algorithm at each step commits to just one subproblem, which results in its **greedy choice** (locally optimal choice)
- Examples: Minimum spanning tree, single-source shortest paths

Algorithmic Techniques

... when I say: "Divide and Conquer"

- An algorithmic approach (not limited to optimization) that splits a problem into sub-problems, solves each sub-problem recursively, and then combines the solutions into a final solution
- E.g., Mergesort splits input array of size n into two arrays of sizes $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$, sorts them, and merges the two sorted lists into a single sorted list in $O(n)$ time
  - Recursion bottoms out for $n = 1$
- Such algorithms often analyzed via recurrence relations

Proof Techniques

… when I say: "Proof by Contradiction"

- A proof technique in which we assume the opposite (negation) of the premise to be proved and then arrive at a contradiction of some other assumption
- If we are trying to prove premise P, we assume for sake of contradiction $\neg P$ and conclude something we know is false
  - If we argue $\neg P \Rightarrow$ false, then $\neg P$ must be false and P must be true
- E.g., to prove there is no greatest even integer:
  - Assume for sake of contradiction there exists a greatest even integer N

    $$\Rightarrow \forall \text{ even integers } n, \text{ we have } N \geq n \tag{1}$$
  - But $M = N + 2$ is an even integer since it's the sum of two even integers, and $M > N$
  - Therefore, our conclusion (1) is false, so our negated premise is false, so our original premise is true $\square$

Proof Techniques

... when I say: "Proof by Induction"

- A proof technique (typically applied to situations involving non-negative integers) in which we prove a **base case** followed by the **inductive step**
- E.g., prove $S_n = \sum_{i=1}^{n} i = n(n+1)/2$
  - **Base case** $(n = 1)$: $S_1 = 1 = n(n+1)/2$
  - **Inductive step**: Assume holds for n and prove it holds for $n+1$:

$$S_{n+1} = S_n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1) + 2n + 2}{2}$$

$$= \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}$$

$\square$

- Useful for proving **invariants** in algorithms, where some property **always** holds at **every** step, and therefore at the final step

Proof Techniques

… when I say: "Proof by Construction"

- A proof technique often used to prove **existence** of something by directly **constructing** it
- E.g., prove that if $a < b$ then there exists a real number c such that $a < c < b$
  - Set $c = (a + b)/2$ (always exists in $\mathbb{R}$)
  - Since $c - a = (a + b - 2a)/2 = (b - a)/2 > 0$ and $b - c = (2b - a - b)/2 = (b - a)/2 > 0$, we have constructed a c such that $a < c < b$ $\square$
- We will use this extensively when we study **NP-completeness**

Proof Techniques

… when I say: “Proof by Contrapositive”

- Recall that $P \Rightarrow Q$ is logically equivalent to $\neg Q \Rightarrow \neg P$ via contraposition (compare truth tables to convince yourself)
- E.g., prove that if x^2 is even, then x is even
  - Contrapositive says: If x is not even, then x^2 is not even
  - This is easily shown true since x is odd, and the product of two odd numbers is odd
  - Since contrapositive is true, original premise is true $\square$
- Very helpful when proving $P \Leftrightarrow Q$ (“P if and only if Q”) since we could prove:
  - $P \Rightarrow Q$ and $\neg P \Rightarrow \neg Q$ **OR**
  - $P \Rightarrow Q$ and $Q \Rightarrow P$ (often simpler)
- We will use this extensively when we study **NP-completeness**

Conclusion

- This was a deliberately brief overview of concepts you should already know
- I expect you to understand it well during lectures, homeworks, and exams
- **It is all covered in depth in the textbook and other resources!**