Computer Science & Engineering 423/823 Design and Analysis of Algorithms

Lecture 09 — Dynamic Programming (Chapter 15)

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(Adapted from Vinodchandran N. Variyam)

Introduction

- Dynamic programming is a technique for solving optimization problems
- Key element: Decompose a problem into **subproblems**, solve them recursively, and then combine the solutions into a final (optimal) solution
- Important component: There are typically an exponential number of subproblems to solve, but many of them overlap

$\Rightarrow$ Can re-use the solutions rather than re-solving them

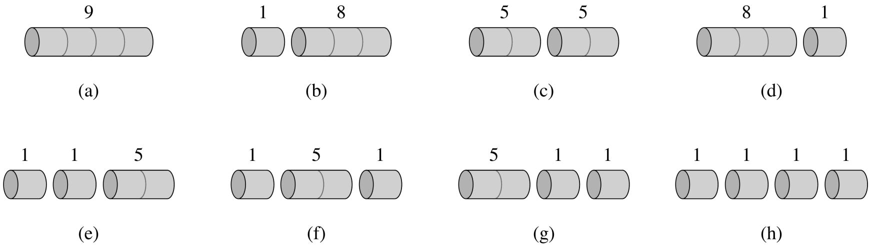
- Number of distinct subproblems is polynomial

Rod Cutting

- A company has a rod of length n and wants to cut it into smaller rods to maximize profit
- Have a table telling how much they get for rods of various lengths: A rod of length i has price p_i
- The cuts themselves are free, so profit is based solely on the prices charged for of the rods
- If cuts only occur at integral boundaries $1, 2, \ldots, n-1$, then can make or not make a cut at each of $n-1$ positions, so total number of possible solutions is 2^{n-1}

Example: Rod Cutting (2)

i	1	2	3	4	5	6	7	8	9	10
p_i	1	5	8	9	10	17	17	20	24	30



Example: Rod Cutting (3)

- Given a rod of length n, want to find a set of cuts into lengths $i_1, \ldots, i_k$ (where $i_1 + \cdots + i_k = n$) and $r_n = p_{i_1} + \cdots + p_{i_k}$ is maximized
- For a specific value of n, can either make no cuts (revenue $= p_n$) or make a cut at some position i, then optimally solve the problem for lengths i and $n - i$:

$$r_n = \max\left(p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_i + r_{n-i}, \ldots, r_{n-1} + r_1\right)$$

- Notice that this problem has the **optimal substructure property**, in that an optimal solution is made up of optimal solutions to subproblems
  - Can find optimal solution if we consider all possible subproblems
- Alternative formulation: Don't further cut the first segment:

$$r_n = \max_{1 \le i \le n} \left(p_i + r_{n-i}\right)$$

Recursive Cut-Rod(p, n)

```
if n == 0 then
    return 0
q = −∞
for i = 1 to n do
    q = max(q, p[i] + Cut-Rod(p, n − i))
end
return q
```

What is the time complexity?

Time Complexity

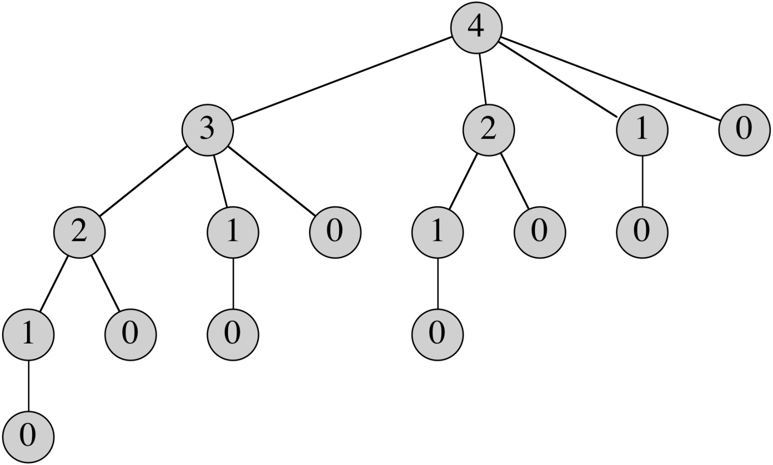
- Let $T(n)$ be number of calls to Cut-Rod
- Thus $T(0) = 1$ and, based on the **for** loop,

$$T(n) = 1 + \sum_{j=0}^{n-1} T(j) = 2^n$$

- Why exponential? Cut-Rod exploits the optimal substructure property, but repeats work on these subproblems
- E.g. if the first call is for $n = 4$, then there will be:
  - 1 call to Cut-Rod(4)
  - 1 call to Cut-Rod(3)
  - 2 calls to Cut-Rod(2)
  - 4 calls to Cut-Rod(1)
  - 8 calls to Cut-Rod(0)

Time Complexity (2)

Recursion Tree for $n = 4$



Dynamic Programming Algorithm

- Can save time dramatically by remembering results from prior calls
- Two general approaches:
  1. **Top-down with memoization:** Run the recursive algorithm as defined earlier, but before recursive call, check to see if the calculation has already been done and **memoized**
  2. **Bottom-up**: Fill in results for "small" subproblems first, then use these to fill in table for "larger" ones
- Typically have the same asymptotic running time

Memoized-Cut-Rod-Aux(p, n, r)

```
if r[n] ≥ 0 then
    return r[n]                // r initialized to all −∞
if n == 0 then
    q = 0
else
    q = −∞
    for i = 1 to n do
        q = max(q, p[i] + MEMOIZED-CUT-ROD-AUX(p, n − i, r))
    end
    r[n] = q
return q
```

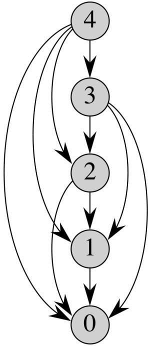
Bottom-Up-Cut-Rod(p, n)

```
Allocate r[0...n]
r[0] = 0
for j = 1 to n do
    q = −∞
    for i = 1 to j do
        q = max (q, p[i] + r[j − i])
    end
    r[j] = q
end
return r[n]
```

First solves for $n = 0$, then for $n = 1$ in terms of $r[0]$, then for $n = 2$ in terms of $r[0]$ and $r[1]$, etc.

Time Complexity

Subproblem graph for $n = 4$



Both algorithms take linear time to solve for each value of n, so total time complexity is $\Theta(n^2)$

Reconstructing a Solution

- If interested in the set of cuts for an optimal solution as well as the revenue it generates, just keep track of the choice made to optimize each subproblem
- Will add a second array s, which keeps track of the optimal size of the first piece cut in each subproblem

Extended-Bottom-Up-Cut-Rod(p, n)


```
Allocate r[0...n] and s[0...n]
r[0] = 0
for j = 1 to n do
    q = −∞
    for i = 1 to j do
        if q < p[i] + r[j − i] then
            q = p[i] + r[j − i]
            s[j] = i

    end
    r[j] = q
end
return r, s
```

Print-Cut-Rod-Solution(p, n)

$(r, s) = \text{Extended-Bottom-Up-Cut-Rod}(p, n)$

```
while n > 0 do
    print s[n]
    n = n − s[n]
end
```

Example:

i	0	1	2	3	4	5	6	7	8	9	10
$r[i]$	0	1	5	8	10	13	17	18	22	25	30
$s[i]$	0	1	2	3	2	2	6	1	2	3	10

If $n = 10$, optimal solution is no cut; if $n = 7$, then cut once to get segments of sizes 1 and 6

Matrix-Chain Multiplication

- Given a chain of matrices $\langle A_1, \ldots, A_n \rangle$, goal is to compute their product $A_1 \cdots A_n$
- This operation is associative, so can sequence the multiplications in multiple ways and get the same result
- Can cause dramatic changes in number of operations required
- Multiplying a $p \times q$ matrix by a $q \times r$ matrix requires pqr steps and yields a $p \times r$ matrix for future multiplications
- E.g. Let A_1 be 10×100, A_2 be 100×5, and A_3 be 5×50
    1. Computing $((A_1 A_2) A_3)$ requires $10 \cdot 100 \cdot 5 = 5000$ steps to compute $(A_1 A_2)$ (yielding a 10×5), and then $10 \cdot 5 \cdot 50 = 2500$ steps to finish, for a total of 7500
    2. Computing $(A_1 (A_2 A_3))$ requires $100 \cdot 5 \cdot 50 = 25000$ steps to compute $(A_2 A_3)$ (yielding a 100×50), and then $10 \cdot 100 \cdot 50 = 50000$ steps to finish, for a total of 75000

Matrix-Chain Multiplication (2)

- The **matrix-chain multiplication problem** is to take a chain $\langle A_1, \ldots, A_n \rangle$ of n matrices, where matrix i has dimension $p_{i-1} \times p_i$, and fully parenthesize the product $A_1 \cdots A_n$ so that the number of scalar multiplications is minimized
- Brute force solution is infeasible, since its time complexity is $\Omega\left(4^n / n^{3/2}\right)$
- Will follow 4-step procedure for dynamic programming:
  1. Characterize the structure of an optimal solution
  2. Recursively define the value of an optimal solution
  3. Compute the value of an optimal solution
  4. Construct an optimal solution from computed information

Characterizing the Structure of an Optimal Solution

- Let $A_{i...j}$ be the matrix from the product $A_i A_{i+1} \cdots A_j$
- To compute $A_{i...j}$, must split the product and compute $A_{i...k}$ and $A_{k+1...j}$ for some integer k, then multiply the two together
- Cost is the cost of computing each subproduct plus cost of multiplying the two results
- Say that in an optimal parenthesization, the optimal split for $A_i A_{i+1} \cdots A_j$ is at k
- Then in an optimal solution for $A_i A_{i+1} \cdots A_j$, the parenthisization of $A_i \cdots A_k$ is itself optimal for the subchain $A_i \cdots A_k$ (if not, then we could do better for the larger chain)
- Similar argument for $A_{k+1} \cdots A_j$
- Thus if we make the right choice for k and then optimally solve the subproblems recursively, we'll end up with an optimal solution
- Since we don't know optimal k, we'll try them all

Recursively Defining the Value of an Optimal Solution

- Define $m[i, j]$ as minimum number of scalar multiplications needed to compute $A_{i...j}$
- (What entry in the m table will be our final answer?)
- Computing $m[i, j]$:
  1. If $i = j$, then no operations needed and $m[i, i] = 0$ for all i
  2. If $i < j$ and we split at k, then optimal number of operations needed is the optimal number for computing $A_{i...k}$ and $A_{k+1...j}$, plus the number to multiply them:

  $$m[i, j] = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$$

  3. Since we don't know k, we'll try all possible values:

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min_{i \le k < j}\{m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\} & \text{if } i < j \end{cases}$$

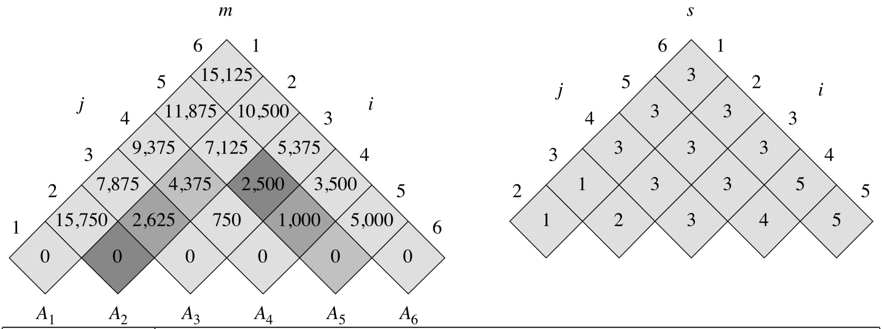
- To track the optimal solution itself, define $s[i, j]$ to be the value of k used at each split

Computing the Value of an Optimal Solution

- As with the rod cutting problem, many of the subproblems we've defined will overlap
- Exploiting overlap allows us to solve only $\Theta(n^2)$ problems (one problem for each (i, j) pair), as opposed to exponential
- We'll do a bottom-up implementation, based on chain length
- Chains of length 1 are trivially solved ($m[i, i] = 0$ for all i)
- Then solve chains of length 2, 3, etc., up to length n
- Linear time to solve each problem, quadratic number of problems, yields $O(n^3)$ total time

Matrix-Chain-Order(p, n)


```
allocate m[1...n, 1...n] and s[1...n, 1...n]
initialize m[i, i] = 0 ∀ 1 ≤ i ≤ n
for ℓ = 2 to n do
    for i = 1 to n − ℓ + 1 do
        j = i + ℓ − 1
        m[i, j] = ∞
        for k = i to j − 1 do
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j] then
                m[i, j] = q
                s[i, j] = k

        end
    end
end
return (m, s)
```



matrix	A_1	A_2	A_3	A_4	A_5	A_6
dimension	30×35	35×15	15×5	5×10	10×20	20×25

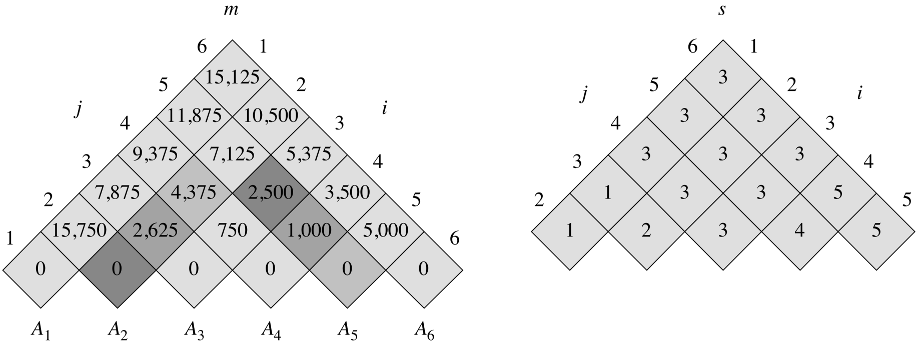
Constructing an Optimal Solution from Computed Information

- Cost of optimal parenthesization is stored in $m[1, n]$
- First split in optimal parenthesization is between $s[1, n]$ and $s[1, n] + 1$
- Descending recursively, next splits are between $s[1, s[1, n]]$ and $s[1, s[1, n]] + 1$ for left side and between $s[s[1, n] + 1, n]$ and $s[s[1, n] + 1, n] + 1$ for right side
- and so on...

Print-Optimal-Parens(s, i, j)


```
if i == j then
    print "A"_i
else
    print "("
    PRINT-OPTIMAL-PARENS(s, i, s[i, j])
    PRINT-OPTIMAL-PARENS(s, s[i, j] + 1, j)
    print ")"
```

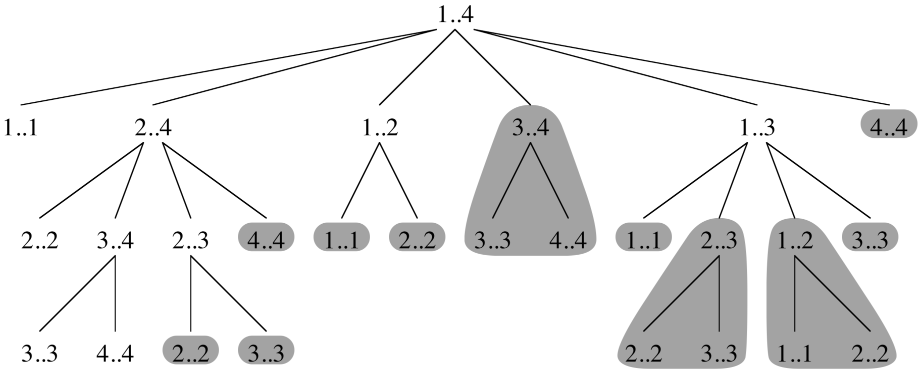
Constructing an Optimal Solution from Computed Information (3)



Optimal parenthesization: $((A_1(A_2A_3))((A_4A_5)A_6))$

Example of How Subproblems Overlap

Entire subtrees overlap:



See Section 15.3 for more on optimal substructure and overlapping subproblems

Longest Common Subsequence

- Sequence $Z = \langle z_1, z_2, \ldots, z_k \rangle$ is a **subsequence** of another sequence $X = \langle x_1, x_2, \ldots, x_m \rangle$ if there is a strictly increasing sequence $\langle i_1, \ldots, i_k \rangle$ of indices of X such that for all $j = 1, \ldots, k$, $x_{i_j} = z_j$
- I.e. as one reads through Z, one can find a match to each symbol of Z in X, in order (though not necessarily contiguous)
- E.g. $Z = \langle B, C, D, B \rangle$ is a subsequence of $X = \langle A, B, C, B, D, A, B \rangle$ since $z_1 = x_2$, $z_2 = x_3$, $z_3 = x_5$, and $z_4 = x_7$
- Z is a **common subsequence** of X and Y if it is a subsequence of both
- The goal of the **longest common subsequence problem** is to find a maximum-length common subsequence (LCS) of sequences $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$

Characterizing the Structure of an Optimal Solution

- Given sequence $X = \langle x_1, \ldots, x_m \rangle$, the ith **prefix** of X is $X_i = \langle x_1, \ldots, x_i \rangle$
- **Theorem** If $X = \langle x_1, \ldots, x_m \rangle$ and $Y = \langle y_1, \ldots, y_n \rangle$ have LCS $Z = \langle z_1, \ldots, z_k \rangle$, then
  1. $x_m = y_n \Rightarrow z_k = x_m = y_n$ and Z_{k-1} is LCS of X_{m-1} and Y_{n-1}
     - If $z_k \neq x_m$, can lengthen Z, $\Rightarrow$ contradiction
     - If Z_{k-1} not LCS of X_{m-1} and Y_{n-1}, then a longer CS of X_{m-1} and Y_{n-1} could have x_m appended to it to get CS of X and Y that is longer than Z, $\Rightarrow$ contradiction
  2. If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is an LCS of X_{m-1} and Y
     - If $z_k \neq x_m$, then Z is a CS of X_{m-1} and Y. Any CS of X_{m-1} and Y that is longer than Z would also be a longer CS for X and Y, $\Rightarrow$ contradiction
  3. If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is an LCS of X and Y_{n-1}
     - Similar argument to (2)

Recursively Defining the Value of an Optimal Solution

- The theorem implies the kinds of subproblems that we'll investigate to find LCS of $X = \langle x_1, \ldots, x_m \rangle$ and $Y = \langle y_1, \ldots, y_n \rangle$
- If $x_m = y_n$, then find LCS of X_{m-1} and Y_{n-1} and append x_m $(= y_n)$ to it
- If $x_m \neq y_n$, then find LCS of X and Y_{n-1} and find LCS of X_{m-1} and Y and identify the longest one
- Let $c[i, j]$ = length of LCS of X_i and Y_j

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j \\ \max\left(c[i, j-1], c[i-1, j]\right) & \text{if } i, j > 0 \text{ and } x_i \neq y_j \end{cases}$$

LCS-Length(X, Y, m, n)

```
allocate b[1...m, 1...n] and c[0...m, 0...n]
initialize c[i,0] = 0 and c[0,j] = 0 ∀ 0 ≤ i ≤ m and 0 ≤ j ≤ n
for i = 1 to m do
    for j = 1 to n do
        if x_i == y_j then
            c[i,j] = c[i-1, j-1] + 1
            b[i,j] = "↖"
        else if c[i-1, j] ≥ c[i, j-1] then
            c[i,j] = c[i-1, j]
            b[i,j] = "↑"
        else
            c[i,j] = c[i, j-1]
            b[i,j] = "←"

    end
end
return (c, b)
```

What is the time complexity?

Computing the Value of an Optimal Solution (2)

$X = \langle A, B, C, B, D, A, B \rangle$, $Y = \langle B, D, C, A, B, A \rangle$

i \ j		0	1	2	3	4	5	6
		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3	C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

Constructing an Optimal Solution from Computed Information

- Length of LCS is stored in $c[m, n]$
- To print LCS, start at $b[m, n]$ and follow arrows until in row or column 0
- If in cell (i, j) on this path, when $x_i = y_j$ (i.e. when arrow is "$\nwarrow$"), print x_i as part of the LCS
- This will print LCS backwards

Print-LCS(b, X, i, j)

if $i == 0$ *or* $j == 0$ **then**
    return
if $b[i, j] ==$ "$\nwarrow$" **then**
    $\textsc{Print-LCS}(b, X, i-1, j-1)$
    print x_i
else if $b[i, j] ==$ "$\uparrow$" **then**
    $\textsc{Print-LCS}(b, X, i-1, j)$
else $\textsc{Print-LCS}(b, X, i, j-1)$

What is the time complexity?

Constructing an Optimal Solution from Computed Information (3)

$X = \langle A, B, C, B, D, A, B \rangle$, $Y = \langle B, D, C, A, B, A \rangle$, prints "BCBA"

i	x_i \ j	0 (y_j)	1 (B)	2 (D)	3 (C)	4 (A)	5 (B)	6 (A)
0	x_i	0	0	0	0	0	0	0
1	A	0	↑0	↑0	↑0	↖1	←1	↖1
2	B	0	↖1	←1	←1	↑1	↖2	←2
3	C	0	↑1	↑1	↖2	←2	↑2	↑2
4	B	0	↖1	↑1	↑2	↑2	↖3	←3
5	D	0	↑1	↖2	↑2	↑2	↑3	↑3
6	A	0	↑1	↑2	↑2	↖3	↑3	↖4
7	B	0	↖1	↑2	↑2	↑3	↖4	↑4

Optimal Binary Search Trees

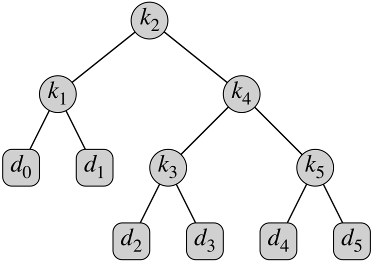
- Goal is to construct binary search trees such that most frequently sought values are near the root, thus minimizing expected search time
- Given a sequence $K = \langle k_1, \ldots, k_n \rangle$ of n distinct keys in sorted order
- Key k_i has probability p_i that it will be sought on a particular search
- To handle searches for values not in K, have $n + 1$ *dummy keys* $d_0, d_1, \ldots, d_n$ to serve as the tree's leaves
- Dummy key d_i will be reached with probability q_i
- If $\mathrm{depth}_T(k_i)$ is distance from root of k_i in tree T, then expected search cost of T is

$$1 + \sum_{i=1}^{n} p_i \, \mathrm{depth}_T(k_i) + \sum_{i=0}^{n} q_i \, \mathrm{depth}_T(d_i)$$

- An **optimal binary search tree** is one with minimum expected search cost

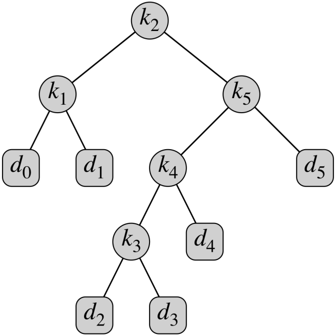
Optimal Binary Search Trees (2)

i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10



(a)

expected cost $= 2.80$



(b)

expected cost $= 2.75$ (optimal)

Characterizing the Structure of an Optimal Solution

- Observation: Since K is sorted and dummy keys interspersed in order, any subtree of a BST must contain keys in a contiguous range $k_i, \ldots, k_j$ and have leaves $d_{i-1}, \ldots, d_j$
- Thus, if an optimal BST T has a subtree T' over keys $k_i, \ldots, k_j$, then T' is optimal for the subproblem consisting of only the keys $k_i, \ldots, k_j$
  - If T' weren't optimal, then a lower-cost subtree could replace T' in T, $\Rightarrow$ contradiction
- Given keys $k_i, \ldots, k_j$, say that its optimal BST roots at k_r for some $i \leq r \leq j$
- Thus if we make right choice for k_r and optimally solve the problem for $k_i, \ldots, k_{r-1}$ (with dummy keys $d_{i-1}, \ldots, d_{r-1}$) and the problem for $k_{r+1}, \ldots, k_j$ (with dummy keys $d_r, \ldots, d_j$), we'll end up with an optimal solution
- Since we don't know optimal k_r, we'll try them all

Recursively Defining the Value of an Optimal Solution

- Define $e[i,j]$ as the expected cost of searching an optimal BST built on keys $k_i, \ldots, k_j$
- If $j = i - 1$, then there is only the dummy key d_{i-1}, so $e[i, i-1] = q_{i-1}$
- If $j \geq i$, then choose root k_r from $k_i, \ldots, k_j$ and optimally solve subproblems $k_i, \ldots, k_{r-1}$ and $k_{r+1}, \ldots, k_j$
- When combining the optimal trees from subproblems and making them children of k_r, we increase their depth by 1, which increases the cost of each by the sum of the probabilities of its nodes
- Define $w(i,j) = \sum_{\ell=i}^{j} p_\ell + \sum_{\ell=i-1}^{j} q_\ell$ as the sum of probabilities of the nodes in the subtree built on $k_i, \ldots, k_j$, and get

$$e[i,j] = p_r + (e[i, r-1] + w(i, r-1)) + (e[r+1, j] + w(r+1, j))$$

Recursively Defining the Value of an Optimal Solution (2)

- Note that
$$w(i,j) = w(i,r-1) + p_r + w(r+1,j)$$
- Thus we can condense the equation to
$e[i,j] = e[i,r-1] + e[r+1,j] + w(i,j)$
- Finally, since we don't know what k_r should be, we try them all:

$$e[i,j] = \begin{cases} q_{i-1} & \text{if } j = i-1 \\ \min_{i \le r \le j}\{e[i,r-1] + e[r+1,j] + w(i,j)\} & \text{if } i \le j \end{cases}$$

- Will also maintain table $root[i,j]$ = index r for which k_r is root of an optimal BST on keys $k_i, \ldots, k_j$

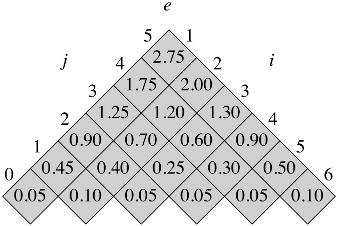
Optimal-BST(p, q, n)

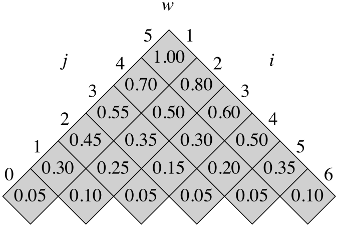
```
allocate e[1...n+1, 0...n], w[1...n+1, 0...n], and
root[1...n, 1...n]
initialize e[i, i−1] = w[i, i−1] = q_{i−1} ∀ 1 ≤ i ≤ n+1
for ℓ = 1 to n do
    for i = 1 to n − ℓ + 1 do
        j = i + ℓ − 1
        e[i, j] = ∞
        w[i, j] = w[i, j−1] + p_j + q_j
        for r = i to j do
            t = e[i, r−1] + e[r+1, j] + w[i, j]
            if t < e[i, j] then
                e[i, j] = t
                root[i, j] = r

        end
    end
end
return (e, root)
```

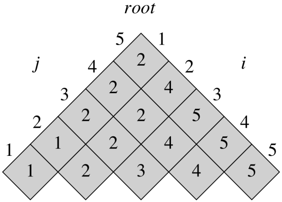
What is the time complexity?

Computing the Value of an Optimal Solution (2)

i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10







Constructing an Optimal Solution from Computed Information

In-class exercise

Write pseudocode for the procedure $\textsc{Construct-Optimal-BST}(root)$ that, given the table $root$, outputs the structure of an optimal binary search tree. It should output text like:

k_2 is the root
k_1 is the left child of k_2
d_0 is the left child of k_1
d_1 is the right child of k_1
k_5 is the right child of k_2
k_4 is the left child of k_5
k_3 is the left child of k_4
… and so on