CSCE423/823

Computer Science & Engineering 423/823 Design and Analysis of Algorithms

Lecture 08 — NP-Completeness (Chapter 34)

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(Adapted from Vinodchandran N. Variyam)

Introduction

- So far, we have focused on problems with "efficient" algorithms
- I.e. problems with algorithms that run in polynomial time: $O(n^c)$ for some constant c
  - Side note: We call it efficient even if c is large, since it is likely that another, even more efficient, algorithm exists
- But, for some problems, the fastest known algorithms require time that is **superpolynomial**
  - Includes sub-exponential time (e.g. $2^{n^{1/3}}$), exponential time (e.g. 2^n), doubly exponential time (e.g. 2^{2^n}), etc.
  - There are even problems that cannot be solved in *any* amount of time (e.g. the "halting problem")

P vs. NP

- Our focus will be on the **complexity classes** called P and NP
- Centers on the notion of a **Turing machine** (TM), which is a finite state machine with an infinitely long tape for storage
  - Anything a computer can do, a TM can do, and vice-versa
  - More on this in CSCE 428/828 and CSCE 424/824
- P = "deterministic polynomial time" = the set of problems that can be solved by a deterministic TM (deterministic algorithm) in polynomial time
- NP = "nondeterministic polynomial time" = the set of problems that can be solved by a nondeterministic TM in polynomial time
  - Can loosely think of a nondeterministic TM as one that can explore many, many possible paths of computation at once
  - Equivalently, NP is the set of problems whose solutions, if given, can be **verified** in polynomial time

P vs. NP Example

- Problem HAM-CYCLE: Does a graph $G = (V, E)$ contain a **hamiltonian cycle**, i.e. a simple cycle that visits every vertex in V exactly once?
  - This problem is in NP, since if we were given a specific G plus the answer to the question plus a **certificate**, we can verify a "yes" answer in polynomial time using the certificate
  - What would be an appropriate certificate?
  - Not known if HAM-CYCLE $\in$ P

P vs. NP Example (2)

- Problem EULER: Does a directed graph $G = (V, E)$ contain an **Euler tour**, i.e. a cycle that visits every edge in E exactly once and can visit vertices multiple times?
  - This problem is in P, since we can answer the question in polynomial time by checking if each vertex's in-degree equals its out-degree
  - Does that mean that the problem is also in NP? If so, what is the certificate?

NP-Completeness

- Any problem in P is also in NP, since if we can efficently solve the problem, we get the poly-time verification for free
  $\Rightarrow$ P $\subseteq$ NP
- Not known if P $\subset$ NP, i.e. unknown if there a problem in NP that's not in P
- A subset of the problems in NP is the set of **NP-complete** (NPC) problems
  - Every problem in NPC is at least as hard as all others in NP
  - These problems are believed to be intractable (no efficient algorithm), but not yet proven to be so
  - If any NPC problem is in P, then P = NP and life is glorious $\smile$

Proving NP-Completeness

- Thus, if we prove that a problem is NPC, we can tell our boss that we cannot find an efficient algorithm and should take a different approach
  - E.g. approximation algorithm, heuristic approach
- How do we prove that a problem A is NPC?
  1. Prove that $A \in$ NP by finding certificate
  2. Show that A is as hard as any other NP problem by showing that if we can efficiently solve A then we can efficiently solve all problems in NP
- First step is usually easy, but second looks difficult
- Fortunately, part of the work has been done for us ...

Reductions

- We will use the idea of a **reduction** of one problem to another to prove how hard it is
- A reduction takes an instance of one problem A and transforms it to an instance of another problem B in such a way that a solution to the instance of B yields a solution to the instance of A
- Example 1: How did we solve the bipartite matching problem?
- Example 2: How did we solve the topological sort problem?
- Time complexity of reduction-based algorithm for A is the time for the reduction to B plus the time to solve the instance of B

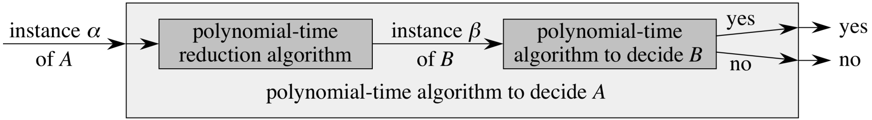
Decision Problems

- Before we go further into reductions, we simplify our lives by focusing on **decision problems**
- In a decision problem, the only output of an algorithm is an answer "yes" or "no"
- I.e. we're not asked for a shortest path or a hamiltonian cycle, etc.
- Not as restrictive as it may seem: Rather than asking for the weight of a shortest path from i to j, just ask if there exists a path from i to j with weight at most k
- Such decision versions of *optimization problems* are no harder than the original optimization problem (why?), so if we show the decision version is hard, then so is the optimization version
- Decision versions are especially convenient when thinking in terms of languages and the Turing machines that accept/reject them

Reductions (2)

- What is a reduction in the NPC sense?
- Start with two problems A and B, and we want to show that problem B is at least as hard as A
- Will **reduce** A to B via a **polynomial-time reduction** by transforming *any* instance α of A to *some* instance β of B such that
  1. The transformation must take polynomial time (since we're talking about hardness in the sense of efficient vs. inefficient algorithms)
  2. The answer for α is "yes" if and only if the answer for β is "yes"
- If such a reduction exists, then B is at least as hard as A since if an efficient algorithm exists for B, we can solve any instance of A in polynomial time
- Notation: $A \leq_{\mathrm{P}} B$, which reads as "A is no harder to solve than B, modulo polynomial time reductions"

Reductions (3)



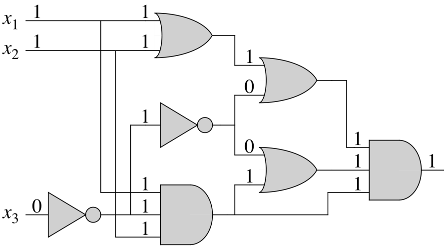
Reductions (4)

- But if we want to prove that a problem B is NPC, do we have to reduce to it *every* problem in NP?
- No we don't:
  - If another problem A is known to be NPC, then we know that any problem in NP reduces to it
  - If we reduce A to B, then any problem in NP can reduce to B via its reduction to A followed by A's reduction to B
  - We then can call B an **NP-hard** problem, which is NPC if it is also in NP
  - Still need our first NPC problem to use as a basis for our reductions

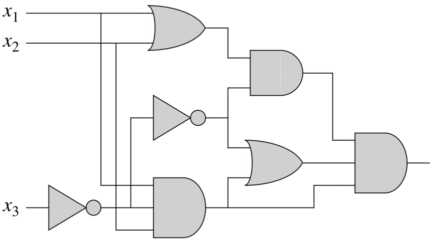
CIRCUIT-SAT

- Our first NPC problem: CIRCUIT-SAT
- An instance is a boolean combinational circuit (no feedback, no memory)
- Question: Is there a **satisfying assignment**, i.e. an assignment of inputs to the circuit that satisfies it (makes its output 1)?

CIRCUIT-SAT (2)



Satisfiable



Unsatisfiable

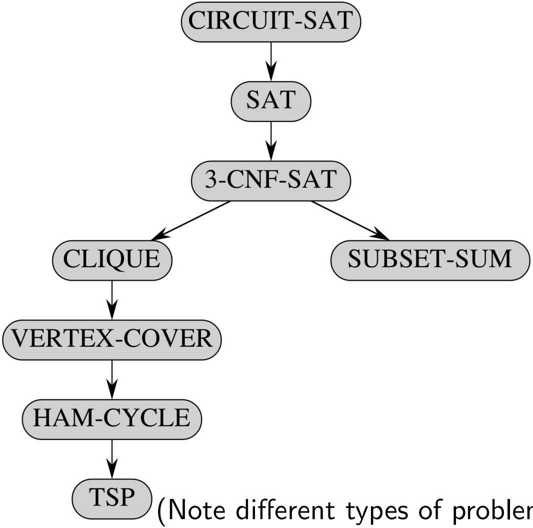
CIRCUIT-SAT (3)

- To prove CIRCUIT-SAT to be NPC, need to show:
  1. CIRCUIT-SAT $\in$ NP; what is its certificate that we can confirm in polynomial time?
  2. That any problem in NP reduces to CIRCUIT-SAT
- We'll skip the NP-hardness proof, save to say that it leverages the existence of an algorithm that verifies certificates for some NP problem

Other NPC Problems

- We'll use the fact that CIRCUIT-SAT is NPC to prove that these other problems are as well:
  - SAT: Does boolean formula ϕ have a satisfying assignment?
  - 3-CNF-SAT: Does 3-CNF formula ϕ have a satisfying assignment?
  - CLIQUE: Does graph G have a clique (complete subgraph) of k vertices?
  - VERTEX-COVER: Does graph G have a vertex cover (set of vertices that touches all edges) of k vertices?
  - HAM-CYCLE: Does graph G have a hamiltonian cycle?
  - TSP: Does complete, weighted graph G have a hamiltonian cycle of total weight $\leq k$?
  - SUBSET-SUM: Is there a subset S' of finite set S of integers that sum to exactly a specific target value t?
- Many more in Garey & Johnson's book, with proofs

Other NPC Problems (2)



(Note different types of problems involved in reductions)

NPC Problem: Formula Satisfiability (SAT)

- Given: A boolean formula ϕ consisting of
  1. n boolean variables $x_1, \ldots, x_n$
  2. m boolean connectives from $\wedge$, $\vee$, $\neg$, $\rightarrow$, and $\leftrightarrow$
  3. Parentheses
- Question: Is there an assignment of boolean values to $x_1, \ldots, x_n$ to make ϕ evaluate to 1?
- E.g.: $\phi = ((x_1 \rightarrow x_2) \vee \neg((\neg x_1 \leftrightarrow x_3) \vee x_4)) \wedge \neg x_2$ has satisfying assignment $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $x_4 = 1$ since

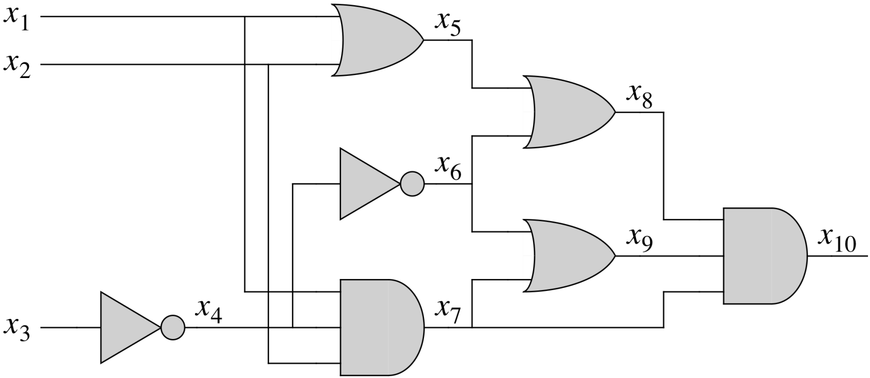
$$\begin{aligned}
\phi &= ((0 \rightarrow 0) \vee \neg((\neg 0 \leftrightarrow 1) \vee 1)) \wedge \neg 0 \\
&= (1 \vee \neg((1 \leftrightarrow 1) \vee 1)) \wedge 1 \\
&= (1 \vee \neg(1 \vee 1)) \wedge 1 \\
&= (1 \vee 0) \wedge 1 \\
&= 1
\end{aligned}$$

SAT is NPC

- SAT is in NP: ϕ's satisfying assignment certifies that the answer is "yes" and this can be easily checked in poly time
- SAT is NP-hard: Will show CIRCUIT-SAT $\leq_{\mathrm{P}}$ SAT by reducing from CIRCUIT-SAT to SAT
- In reduction, need to map *any* instance (circuit) C of CIRCUIT-SAT to *some* instance (formula) ϕ of SAT such that C has a satisfying assignment if and only if ϕ does
- Further, the time to do the mapping must be polynomial in the size of the circuit, implying that ϕ's representation must be polynomially sized

SAT is NPC (2)

Define a variable in ϕ for each wire in C:



SAT is NPC (3)

- Then define a clause of ϕ for each gate that defines the function for that gate:

$$
\begin{aligned}
\phi = x_{10} \quad & \wedge \quad (x_4 \leftrightarrow \neg x_3) \\
& \wedge \quad (x_5 \leftrightarrow (x_1 \vee x_2)) \\
& \wedge \quad (x_6 \leftrightarrow \neg x_4) \\
& \wedge \quad (x_7 \leftrightarrow (x_1 \wedge x_2 \wedge x_4)) \\
& \wedge \quad (x_8 \leftrightarrow (x_5 \vee x_6)) \\
& \wedge \quad (x_9 \leftrightarrow (x_6 \vee x_7)) \\
& \wedge \quad (x_{10} \leftrightarrow (x_7 \wedge x_8 \wedge x_9))
\end{aligned}
$$

SAT is NPC (4)

- Size of ϕ is polynomial in size of C (number of gates and wires)

$\Rightarrow$ If C has a satisfying assignment, then the final output of the circuit is 1 and the value on each internal wire matches the output of the gate that feeds it

  - Thus, ϕ evaluates to 1

$\Leftarrow$ If ϕ has a satisfying assignment, then each of ϕ's clauses is satisfied, which means that each of C's gate's output matches its function applied to its inputs, and the final output is 1

- Since satisfying assignment for $C \Rightarrow$ satisfying assignment for ϕ and vice-versa, we get C has a satisfying assignment if and only if ϕ does

NPC Problem: 3-CNF Satisfiability (3-CNF-SAT)

- Given: A boolean formula that is in 3-conjunctive normal form (3-CNF), which is a conjunction of clauses, each a disjunction of 3 literals, e.g.

$$(x_1 \vee \neg x_1 \vee \neg x_2) \wedge (x_3 \vee x_2 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4) \wedge (x_4 \vee x_5 \vee x_1)$$

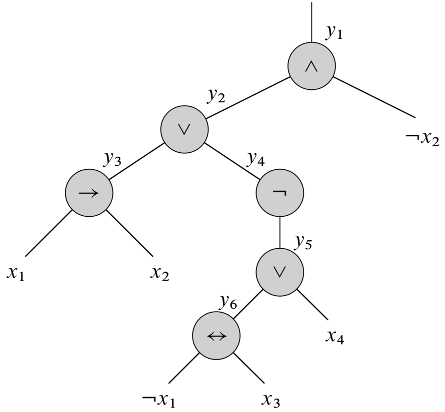
- Question: Is there an assignment of boolean values to $x_1, \ldots, x_n$ to make the formula evaluate to 1?

3-CNF-SAT is NPC

- 3-CNF-SAT is in NP: The satisfying assignment certifies that the answer is "yes" and this can be easily checked in poly time
- 3-CNF-SAT is NP-hard: Will show SAT $\leq_P$ 3-CNF-SAT
- Again, need to map *any* instance ϕ of SAT to *some* instance ϕ''' of 3-CNF-SAT
  1. Parenthesize ϕ and build its *parse tree*, which can be viewed as a circuit
  2. Assign variables to wires in this circuit, as with previous reduction, yielding ϕ', a conjunction of clauses
  3. Use the truth table of each clause ϕ'_i to get its DNF, then convert it to CNF ϕ''_i
  4. Add auxillary variables to each ϕ''_i to get three literals in it, yielding ϕ'''_i
  5. Final CNF formula is $\phi''' = \bigwedge_i \phi'''_i$

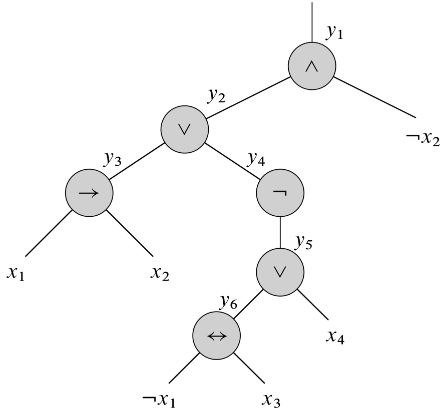
Building the Parse Tree

$$\phi = ((x_1 \to x_2) \vee \neg((\neg x_1 \leftrightarrow x_3) \vee x_4)) \wedge \neg x_2$$



Might need to parenthesize ϕ to put at most two children per node

Assign Variables to wires



$$\phi' = y_1 \wedge (y_1 \leftrightarrow (y_2 \wedge \neg x_2)) \wedge (y_2 \leftrightarrow (y_3 \vee y_4)) \wedge$$
$$(y_3 \leftrightarrow (x_1 \rightarrow x_2)) \wedge (y_4 \leftrightarrow \neg y_5) \wedge (y_5 \leftrightarrow (y_6 \vee x_4)) \wedge (y_6 \leftrightarrow (\neg x_1 \leftrightarrow x_3))$$

Convert Each Clause to CNF

- Consider first clause $\phi'_1 = (y_1 \leftrightarrow (y_2 \wedge \neg x_2))$
- Truth table:

y_1	y_2	x_2	$(y_1 \leftrightarrow (y_2 \wedge \neg x_2))$
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	1

- Can now directly read off DNF of negation:

$$\neg\phi'_1 = (y_1 \wedge y_2 \wedge x_2) \vee (y_1 \wedge \neg y_2 \wedge x_2) \vee (y_1 \wedge \neg y_2 \wedge \neg x_2) \vee (\neg y_1 \wedge y_2 \wedge \neg x_2)$$

- And use DeMorgan's Law to convert it to CNF:

$$\phi''_1 = (\neg y_1 \vee \neg y_2 \vee \neg x_2) \wedge (\neg y_1 \vee y_2 \vee \neg x_2) \wedge (\neg y_1 \vee y_2 \vee x_2) \wedge (y_1 \vee \neg y_2 \vee x_2)$$

Add Auxillary Variables

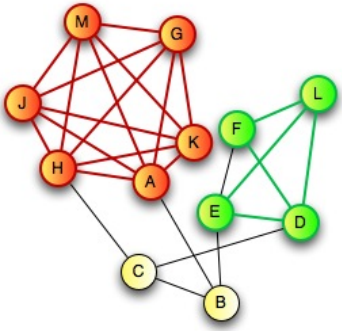
- Based on our construction, $\phi = \phi'' = \bigwedge_i \phi''_i$, where each ϕ''_i is a CNF formula each with at most three literals per clause
- But we need to have *exactly* three per clause!
- Simple fix: For each clause C_i of ϕ'',
  1. If C_i has three distinct literals, add it as a clause in ϕ'''
  2. If $C_i = (\ell_1 \vee \ell_2)$ for distinct literals ℓ_1 and ℓ_2, then add to ϕ''' $(\ell_1 \vee \ell_2 \vee p) \wedge (\ell_1 \vee \ell_2 \vee \neg p)$
  3. If $C_i = (\ell)$, then add to ϕ''' $(\ell \vee p \vee q) \wedge (\ell \vee p \vee \neg q) \wedge (\ell \vee \neg p \vee q) \wedge (\ell \vee \neg p \vee \neg q)$
- p and q are **auxillary variables**, and the combinations in which they're added result in a logically equivalent expression to that of the original clause, regardless of the values of p and q

Proof of Correctness of Reduction

- ϕ has a satisfying assignment iff ϕ''' does
  1. CIRCUIT-SAT reduction to SAT implies satisfiability preserved from ϕ to ϕ'
  2. Use of truth tables and DeMorgan's Law ensures ϕ'' equivalent to ϕ'
  3. Addition of auxillary variables ensures ϕ''' equivalent to ϕ''
- Constructing ϕ''' from ϕ takes polynomial time
  1. ϕ' gets variables from ϕ, plus at most one variable and one clause per operator in ϕ
  2. Each clause in ϕ' has at most 3 variables, so each truth table has at most 8 rows, so each clause in ϕ' yields at most 8 clauses in ϕ''
  3. Since there are only two auxillary variables, each clause in ϕ'' yields at most 4 in ϕ'''
  4. Thus size of ϕ''' is polynomial in size of ϕ, and each step easily done in polynomial time

NPC Problem: Clique Finding (CLIQUE)

- Given: An undirected graph $G = (V, E)$ and value k
- Question: Does G contain a clique (complete subgraph) of size k?



Has a clique of size $k = 6$, but not of size 7

CLIQUE is NPC

- CLIQUE is in NP: A list of vertices in the clique certifies that the answer is "yes" and this can be easily checked in poly time (how?)
- CLIQUE is NP-hard: Will show 3-CNF-SAT $\leq_P$ CLIQUE by mapping *any* instance ϕ of 3-CNF-SAT to *some* instance $\langle G, k \rangle$ of CLIQUE
  - Seems strange to reduce a boolean formula to a graph, but we will show that ϕ has a satisfying assignment iff G has a clique of size k
  - Caveat: the reduction merely preserves the iff relationship; it does not try to directly solve either problem, nor does it assume it knows what the answer is

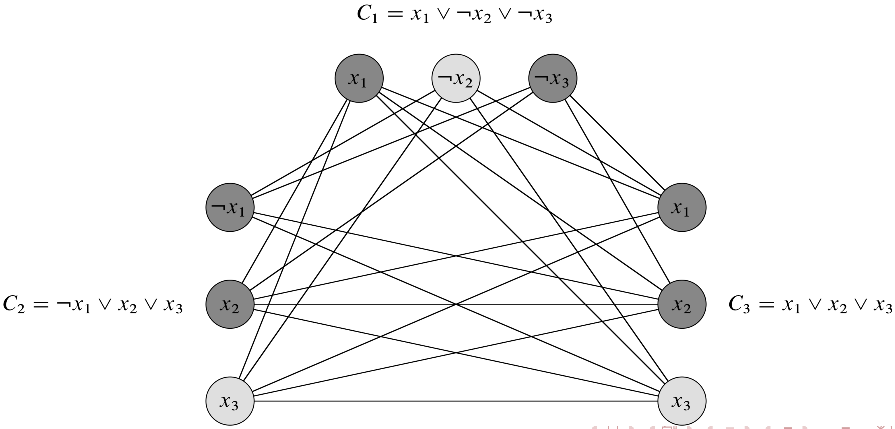
The Reduction

- Let $\phi = C_1 \wedge \cdots \wedge C_k$ be a 3-CNF formula with k clauses
- For each clause $C_r = (\ell_1^r \vee \ell_2^r \vee \ell_3^r)$ put vertices v_1^r, v_2^r, and v_3^r into V
- Add edge (v_i^r, v_j^s) to E if:
    1. $r \neq s$, i.e. v_i^r and v_j^s are in separate triples
    2. ℓ_i^r is not the negation of ℓ_j^s
- Obviously can be done in polynomial time

The Reduction (2)

$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$

Satisfied by $x_2 = 0$, $x_3 = 1$



The Reduction (3)

$\Rightarrow$ If ϕ has a satisfying assignment, then at least one literal in each clause is true

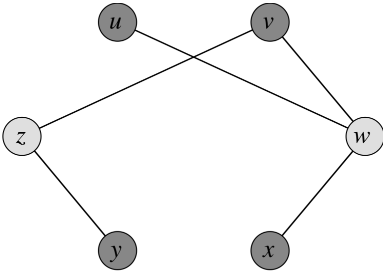
- Picking corresponding vertex from a true literal from each clause yields a set V' of k vertices, each in a distinct triple
- Since each vertex in V' is in a distinct triple and literals that are negations of each other cannot both be true in a satisfying assignment, there is an edge between each pair of vertices in V'
- V' is a clique of size k

$\Leftarrow$ If G has a size-k clique V', can assign 1 to corresponding literal of each vertex in V'

- Each vertex in its own triple, so each clause has a literal set to 1
- Will not try to set both a literal and its negation to 1
- Get a satisfying assignment

NPC Problem: Vertex Cover Finding (VERTEX-COVER)

- A vertex in a graph is said to **cover** all edges incident to it
- A **vertex cover** of a graph is a set of vertices that covers all edges in the graph
- Given: An undirected graph $G = (V, E)$ and value k
- Question: Does G contain a vertex cover of size k?



Has a vertex cover of size $k = 2$, but not of size 1

VERTEX-COVER is NPC

- VERTEX-COVER is in NP: A list of vertices in the vertex cover certifies that the answer is "yes" and this can be easily checked in poly time
- VERTEX-COVER is NP-hard: Will show CLIQUE $\leq_{\mathrm{P}}$ VERTEX-COVER by mapping *any* instance $\langle G, k\rangle$ of CLIQUE to *some* instance $\langle G', k'\rangle$ of VERTEX-COVER
- Reduction is simple: Given instance $\langle G = (V, E), k\rangle$ of CLIQUE, instance of VERTEX-COVER is $\langle \overline{G}, |V| - k\rangle$, where $\overline{G} = (V, \overline{E})$ is G's **complement**:

$$\overline{E} = \{(u, v) : u, v \in V, u \neq v, (u, v) \notin E\}$$

- Easily done in polynomial time

Proof of Correctness

$\Rightarrow$ Assume G has a size-k clique $V' \subseteq V$

- Consider edge $(u, v) \in \overline{E}$
- If it's in $\overline{E}$, then $(u, v) \notin E$, so at least one of u and v (which cover (u, v)) is not in V', so at least one of them is in $V \setminus V'$
- This holds for each edge in $\overline{E}$, so $V \setminus V'$ is a vertex cover of $\overline{G}$ of size $|V| - k$

$\Leftarrow$ Assume $\overline{G}$ has a size-$(|V| - k)$ vertex cover V'

- For each $(u, v) \in \overline{E}$, at least one of u and v is in V'
- By contrapositive, if $u, v \notin V'$, then $(u, v) \in E$
- Since every pair of nodes in $V \setminus V'$ has an edge between them, $V \setminus V'$ is a clique of size $|V| - |V'| = k$

NPC Problem: Subset Sum (SUBSET-SUM)

- Given: A finite set S of positive integers and a positive integer **target** t
- Question: Is there a subset $S' \subseteq S$ whose elements sum to t?
- E.g. $S = \{1, 2, 7, 14, 49, 98, 343, 686, 2409, 2793, 16808, 17206, 117705, 117993\}$ and $t = 138457$ has a solution $S' = \{1, 2, 7, 98, 343, 686, 2409, 17206, 117705\}$

SUBSET-SUM is NPC

- SUBSET-SUM is in NP: The subset S' certifies that the answer is "yes" and this can be easily checked in poly time
- SUBSET-SUM is NP-hard: Will show 3-CNF-SAT $\leq_{\mathrm{P}}$ SUBSET-SUM by mapping *any* instance ϕ of 3-CNF-SAT to *some* instance $\langle S, t \rangle$ of SUBSET-SUM
- Make two reasonable assumptions about ϕ:
  1. No clause contains both a variable and its negation
  2. Each variable appears in at least one clause

The Reduction

- Let ϕ have k clauses $C_1, \ldots, C_k$ over n variables $x_1, \ldots, x_n$
- Reduction creates two numbers in S for each variable x_i and two numbers for each clause C_j
- Each number has $n + k$ digits, the most significant n tied to variables and least significant k tied to clauses
  1. Target t has a 1 in each digit tied to a variable and a 4 in each digit tied to a clause
  2. For each x_i, S contains integers v_i and v_i', each with a 1 in x_i's digit and 0 for other variables. Put a 1 in C_j's digit for v_i if x_i in C_j, and a 1 in C_j's digit for v_i' if $\neg x_i$ in C_j
  3. For each C_j, S contains integers s_j and s_j', where s_j has a 1 in C_j's digit and 0 elsewhere, and s_j' has a 2 in C_j's digit and 0 elsewhere
- Greatest sum of any digit is 6, so no carries when summing integers
- Can be done in polynomial time

The Reduction (2)

$$C_1 = (x_1 \vee \neg x_2 \vee \neg x_3),\ C_2 = (\neg x_1 \vee \neg x_2 \vee \neg x_3),$$
$$C_3 = (\neg x_1 \vee \neg x_2 \vee x_3),\ C_4 = (x_1 \vee x_2 \vee x_3)$$

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
v_1	=	1	0	0	1	0	0	1
v'_1	=	1	0	0	0	1	1	0
v_2	=	0	1	0	0	0	0	1
v'_2	=	0	1	0	1	1	1	0
v_3	=	0	0	1	0	0	1	1
v'_3	=	0	0	1	1	1	0	0
s_1	=	0	0	0	1	0	0	0
s'_1	=	0	0	0	2	0	0	0
s_2	=	0	0	0	0	1	0	0
s'_2	=	0	0	0	0	2	0	0
s_3	=	0	0	0	0	0	1	0
s'_3	=	0	0	0	0	0	2	0
s_4	=	0	0	0	0	0	0	1
s'_4	=	0	0	0	0	0	0	2
t	=	1	1	1	4	4	4	4

$$x_1 = 0,\ x_2 = 0,\ x_3 = 1$$

Proof of Correctness

$\Rightarrow$ If $x_i = 1$ in ϕ's satisfying assignment, SUBSET-SUM solution S' will have v_i, otherwise v_i'

- For each variable-based digit, the sum of the elements of S' is 1
- Since each clause is satisfied, each clause contains at least one literal with the value 1, so each clause-based digit sums to 1, 2, or 3
- To match each clause-based digit in t, add in the appropriate subset of **slack variables** s_i and s_i'

Proof of Correctness (2)

$\Leftarrow$ In SUBSET-SUM solution S', for each $i = 1, \ldots, n$, exactly one of v_i and v_i' must be in S', or sum won't match t

- If $v_i \in S'$, set $x_i = 1$ in satisfying assignment, otherwise we have $v_i' \in S'$ and set $x_i = 0$
- To get a sum of 4 in clause-based digit C_j, S' must include a v_i or v_i' value that is 1 in that digit (since slack variables sum to at most 3)
- Thus, if $v_i \in S'$ has a 1 in C_j's position, then x_i is in C_j and we set $x_i = 1$, so C_j is satisfied (similar argument for $v_i' \in S'$ and setting $x_i = 0$)
- This holds for all clauses, so ϕ is satisfied

In-Class Exercise

- OK, everything perfectly clear?
- Want a shot at extra credit?
- Put away your books (keep your notes), split into groups, and get ready for an in-class exercise!