Computer Science & Engineering 423/823
Design and Analysis of Algorithms

Lecture 06 — All-Pairs Shortest Paths (Chapter 25)

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(Adapted from Vinodchandran N. Variyam)

Introduction

- Similar to SSSP, but find shortest paths for all pairs of vertices
- Given a weighted, directed graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$, find $\delta(u, v)$ for all $(u, v) \in V \times V$
- One solution: Run an algorithm for SSSP $|V|$ times, treating each vertex in V as a source
  - If no negative weight edges, use Dijkstra's algorithm, for time complexity of $O(|V|^3 + |V||E|) = O(|V|^3)$ for array implementation, $O(|V||E| \log |V|)$ if heap used
  - If negative weight edges, use Bellman-Ford and get $O(|V|^2|E|)$ time algorithm, which is $O(|V|^4)$ if graph dense
- Can we do better?
  - Matrix multiplication-style algorithm: $\Theta(|V|^3 \log |V|)$
  - Floyd-Warshall algorithm: $\Theta(|V|^3)$
  - Both algorithms handle negative weight edges

Adjacency Matrix Representation

- Will use adjacency matrix representation
- Assume vertices are numbered: $V = \{1, 2, \ldots, n\}$
- Input to our algorithms will be $n \times n$ matrix W:

$$w_{ij} = \begin{cases} 0 & \text{if } i = j \\ \text{weight of edge } (i,j) & \text{if } (i,j) \in E \\ \infty & \text{if } (i,j) \notin E \end{cases}$$

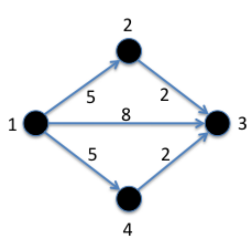
- For now, assume negative weight cycles are absent
- In addition to distance matrices L and D produced by algorithms, can also build *predecessor matrix* Π, where π_{ij} = predecessor of j on a shortest path from i to j, or NIL if $i = j$ or no path exists
  - Well-defined due to optimal substructure property

Print-All-Pairs-Shortest-Path(Π, i, j)

```
if i == j then
    print i
else if π_ij == NIL then
    print "no path from " i " to " j " exists"
else
    PRINT-ALL-PAIRS-SHORTEST-PATH(Π, i, π_ij)
    print j
```


Shortest Paths and Matrix Multiplication

- Will maintain a series of matrices $L^{(m)} = \left(\ell_{ij}^{(m)}\right)$, where $\ell_{ij}^{(m)}$ = the minimum weight of any path from i to j that uses at most m edges
  - Special case: $\ell_{ij}^{(0)} = 0$ if $i = j$, ∞ otherwise



$\ell_{13}^{(0)} = \infty$, $\ell_{13}^{(1)} = 8$, $\ell_{13}^{(2)} = 7$

Recursive Solution

- Can exploit optimal substructure property to get a recursive definition of $\ell_{ij}^{(m)}$
- To follow shortest path from i to j using at most m edges, either:
  1. Take shortest path from i to j using $\leq m - 1$ edges and stay put, or
  2. Take shortest path from i to some k using $\leq m - 1$ edges and traverse edge (k, j)

$$\ell_{ij}^{(m)} = \min\left(\ell_{ij}^{(m-1)}, \min_{1 \leq k \leq n}\left(\ell_{ik}^{(m-1)} + w_{kj}\right)\right)$$

- Since $w_{jj} = 0$ for all j, simplify to

$$\ell_{ij}^{(m)} = \min_{1 \leq k \leq n}\left(\ell_{ik}^{(m-1)} + w_{kj}\right)$$

- If no negative weight cycles, then since all shortest paths have $\leq n - 1$ edges,

$$\delta(i, j) = \ell_{ij}^{(n-1)} = \ell_{ij}^{(n)} = \ell_{ij}^{(n+1)} = \cdots$$

Bottum-Up Computation of L Matrices

- Start with weight matrix W and compute series of matrices $L^{(1)}, L^{(2)}, \ldots, L^{(n-1)}$
- Core of the algorithm is a routine to compute $L^{(m+1)}$ given $L^{(m)}$ and W
- Start with $L^{(1)} = W$, and iteratively compute new L matrices until we get $L^{(n-1)}$
    - Why is $L^{(1)} == W$?
- Can we detect negative-weight cycles with this algorithm? How?

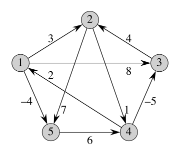
Extend-Shortest-Paths(L, W)

n = number of rows of L // This is $L^{(m)}$
create new $n \times n$ matrix L' // This will be $L^{(m+1)}$
for $i = 1$ *to* n **do**
  for $j = 1$ *to* n **do**
    $\ell'_{ij} = \infty$
    for $k = 1$ *to* n **do**
      $\ell'_{ij} = \min(\ell'_{ij}, \ell_{ik} + w_{kj})$
    end
  end
end
return L'

Slow-All-Pairs-Shortest-Paths(W)

n = number of rows of W
$L^{(1)} = W$
for $m = 2$ *to* $n - 1$ **do**
  $L^{(m)} = \text{EXTEND-SHORTEST-PATHS}(L^{(m-1)}, W)$
end
return $L^{(n-1)}$

Example



$$L^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \quad L^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{pmatrix}$$

$$L^{(3)} = \begin{pmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \quad L^{(4)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

Improving Running Time

- What is time complexity of SLOW-ALL-PAIRS-SHORTEST-PATHS?
- Can we do better?
- Note that if, in EXTEND-SHORTEST-PATHS, we change $+$ to multiplication and $\min$ to $+$, get matrix multiplication of L and W
- If we let $\odot$ represent this "multiplication" operator, then SLOW-ALL-PAIRS-SHORTEST-PATHS computes

$$\begin{aligned} L^{(2)} &= L^{(1)} \odot W &= W^{②}, \\ L^{(3)} &= L^{(2)} \odot W &= W^{③}, \\ &\vdots & \\ L^{(n-1)} &= L^{(n-2)} \odot W &= W^{n \ominus 1} \end{aligned}$$

- Thus, we get $L^{(n-1)}$ by iteratively "multiplying" W via EXTEND-SHORTEST-PATHS

Improving Running Time (2)

- But we don't need every $L^{(m)}$; we only want $L^{(n-1)}$
- E.g. if we want to compute 7^{64}, we could multiply 7 by itself 64 times, or we could square it 6 times
- In our application, once we have a handle on $L^{((n-1)/2)}$, we can immediately get $L^{(n-1)}$ from one call to EXTEND-SHORTEST-PATHS$(L^{((n-1)/2)}, L^{((n-1)/2)})$
- Of course, we can similarly get $L^{((n-1)/2)}$ from "squaring" $L^{((n-1)/4)}$, and so on
- Starting from the beginning, we initialize $L^{(1)} = W$, then compute $L^{(2)} = L^{(1)} \odot L^{(1)}$, $L^{(4)} = L^{(2)} \odot L^{(2)}$, $L^{(8)} = L^{(4)} \odot L^{(4)}$, and so on
- What happens if $n - 1$ is not a power of 2 and we "overshoot" it?
- How many steps of repeated squaring do we need to make?
- What is time complexity of this new algorithm?

Faster-All-Pairs-Shortest-Paths(W)

```
n = number of rows of W
L^(1) = W
m = 1
while m < n − 1 do
    L^(2m) = Extend-Shortest-Paths(L^(m), L^(m))
    m = 2m
end
return L^(m)
```

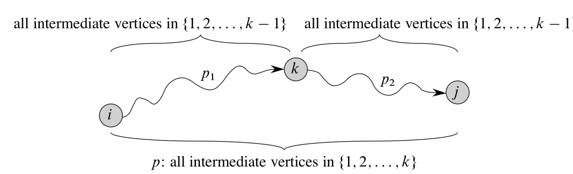

Floyd-Warshall Algorithm

- Shaves the logarithmic factor off of the previous algorithm
- As with previous algorithm, start by assuming that there are no negative weight cycles; can detect negative weight cycles the same way as before
- Considers a different way to decompose shortest paths, based on the notion of an *intermediate vertex*
  - If simple path $p = \langle v_1, v_2, v_3, \ldots, v_{\ell-1}, v_\ell \rangle$, then the set of intermediate vertices is $\{v_2, v_3, \ldots, v_{\ell-1}\}$

Structure of Shortest Path

- Again, let $V = \{1, \ldots, n\}$, and fix $i, j \in V$
- For some $1 \leq k \leq n$, consider set of vertices $V_k = \{1, \ldots, k\}$
- Now consider all paths from i to j whose intermediate vertices come from V_k and let p be the minimum-weight path from them
- Is $k \in p$?
  1. If not, then all intermediate vertices of p are in V_{k-1}, and a SP from i to j based on V_{k-1} is also a SP from i to j based on V_k
  2. If so, then we can decompose p into $i \overset{p_1}{\leadsto} k \overset{p_2}{\leadsto} j$, where p_1 and p_2 are each shortest paths based on V_{k-1}

Structure of Shortest Path (2)



Recursive Solution

- What does this mean?
- It means that the shortest path from i to j based on V_k is either going to be the same as that based on V_{k-1}, or it is going to go through k
- In the latter case, the shortest path from i to j based on V_k is going to be the shortest path from i to k based on V_{k-1}, followed by the shortest path from k to j based on V_{k-1}
- Let matrix $D^{(k)} = \left(d_{ij}^{(k)}\right)$, where $d_{ij}^{(k)}$ = weight of a shortest path from i to j based on V_k:

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1 \end{cases}$$

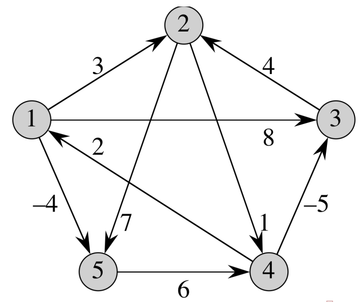
- Since all SPs are based on $V_n = V$, we get $d_{ij}^{(n)} = \delta(i, j)$ for all $i, j \in V$

Floyd-Warshall(W)

```
n = number of rows of W
D^(0) = W
for k = 1 to n do
    for i = 1 to n do
        for j = 1 to n do
            d_ij^(k) = min( d_ij^(k−1), d_ik^(k−1) + d_kj^(k−1) )
        end
    end
end
return D^(n)
```

Floyd-Warshall Example

Split into teams, and simulate Floyd-Warshall on this example:



Transitive Closure

- Used to determine whether paths exist between pairs of vertices
- Given directed, unweighted graph $G = (V, E)$ where $V = \{1, \ldots, n\}$, the *transitive closure* of G is $G^* = (V, E^*)$, where

$$E^* = \{(i, j) : \text{there is a path from } i \text{ to } j \text{ in } G\}$$

- How can we directly apply Floyd-Warshall to find E^*?
- Simpler way: Define matrix T similarly to D:

$$t_{ij}^{(0)} = \begin{cases} 0 & \text{if } i \neq j \text{ and } (i, j) \notin E \\ 1 & \text{if } i = j \text{ or } (i, j) \in E \end{cases}$$

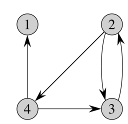
$$t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee \left(t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)} \right)$$

- I.e. you can reach j from i using V_k if you can do so using V_{k-1} or if you can reach k from i and reach j from k, both using V_{k-1}

Transitive-Closure(G)

allocate and initialize $n \times n$ matrix $T^{(0)}$
for $k = 1$ *to* n **do**
    allocate $n \times n$ matrix $T^{(k)}$
    for $i = 1$ *to* n **do**
        for $j = 1$ *to* n **do**
            $t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)}$
        end
    end
end
return $T^{(n)}$

Example



$$T^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad T^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

$$T^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad T^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Analysis

- Like Floyd-Warshall, time complexity is officially $\Theta(n^3)$
- However, use of 0s and 1s exclusively allows implementations to use bitwise operations to speed things up significantly, processing bits in batch, a word at a time
- Also saves space
- Another space saver: Can update the T matrix (and F-W's D matrix) in place rather than allocating a new matrix for each step (Exercise 25.2-4)