Computer Science & Engineering 423/823 Design and Analysis of Algorithms

Lecture 02 — Sorting Lower Bound (Section 8.1)

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(Adapted from Vinodchandran N. Variyam)

Introduction

- Impossibility of algorithms: There are some problems that cannot be solved
  - We'll visit this throughout the semester, especially with NP-completeness
  - Today's example: there does not exist a general-purpose (**comparison-based**) algorithm to sort n elements in time $o(n \log n)$
  - Will show this by proving an $\Omega(n \log n)$ **lower bound** on comparison-based sorting

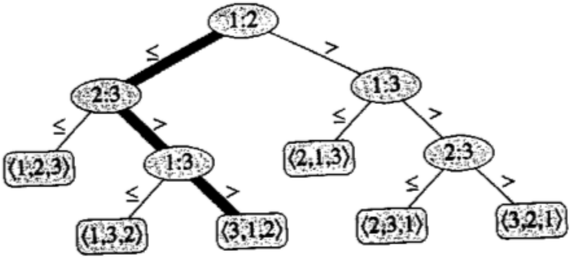
Comparison-Based Sorting Algorithms

- What is a comparison-based sorting algorithm?
    - The sorted order it determines is based **only** on comparisons between the input elements
    - E.g. Insertion Sort, Selection Sort, Mergesort, Quicksort, Heapsort
- What is **not** a comparison-based sorting algorithm?
    - The sorted order it determines is based on additional information, e.g. bounds on the range of input values
    - E.g. Counting Sort, Radix Sort

Decision Trees

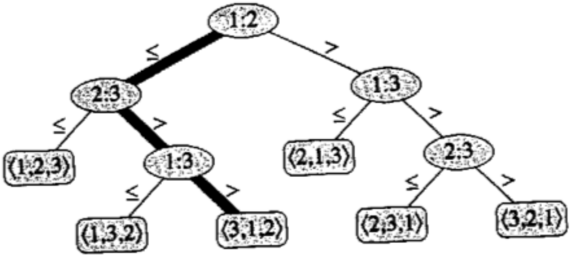
- A **decision tree** is a full binary tree that represents comparisions between elements performed by a particular sorting algorithm operating on a certain-sized input (n elements)
- **Key point:** a tree represents algorithm's behavior on *all possible inputs* of size n
- Each internal node represents one comparison made by algorithm
  - Each node labeled as $i : j$, which represents comparison $A[i] \leq A[j]$
  - If, in the particular input, it is the case that $A[i] \leq A[j]$, then control flow moves to left child, otherwise to the right child
  - Each leaf represents a possible output of the algorithm, which is a permutation of the input
  - All permutations must be in the tree in order for algorithm to work properly

Example for Insertion Sort



- If $n = 3$, Insertion Sort first compares $A[1]$ to $A[2]$
- If $A[1] \leq A[2]$, then compare $A[2]$ to $A[3]$
- If $A[2] > A[3]$, then compare $A[1]$ to $A[3]$
- If $A[1] \leq A[3]$, then sorted order is $A[1], A[3], A[2]$

Example for Insertion Sort (2)



- Example: $A = [7, 8, 4]$
- First compare 7 to 8, then 8 to 4, then 7 to 4
- Output permutation is $\langle 3, 1, 2 \rangle$, which implies sorted order is 4, 7, 8

Proof of Lower Bound

- Length of path from root to output leaf is number of comparisons made by algorithm on that input
- Worst-case number of comparisons is length of longest path ($=$ **height** h)
- Number of leaves in tree is $n!$
- A binary tree of height h has at most 2^h leaves
- Thus we have $2^h \geq n! \geq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$
- Take base-2 logs of both sides to get

$$h \geq \lg \sqrt{2\pi} + (1/2)\lg n + n \lg n - n \lg e = \Omega(n \log n)$$

$\Rightarrow$ **Every** comparison-based sorting algorithm has an input that forces it to make $\Omega(n \log n)$ comparisons

$\Rightarrow$ Mergesort and Heapsort are *asymptotically optimal*