Computer Science & Engineering 423/823
Design and Analysis of Algorithms

Lecture 07 — Maximum Flow (Chapter 26)

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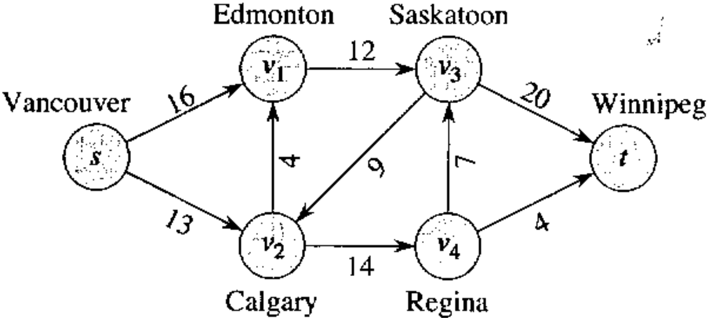
Introduction

- Can use a directed graph as a *flow network* to model:
  - Data through communication networks, water/oil/gas through pipes, assembly lines, etc.
- A *flow network* is a directed graph with two special vertices: *source* s that produces flow and *sink* t that takes in flow
- Each directed edge is a conduit with a certain capacity (e.g. 200 gallons/hour)
- Vertices are conduit junctions
- Except for s and t, flow must be conserved: The flow into a vertex must match the flow out
- *Maximum flow problem:* Given a flow network, determine the maximum amount of flow that can get from s to t
- Other application: Bipartite matching

Flow Networks

- A *flow network* $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative *capacity* $c(u, v) \geq 0$
- If $(u, v) \notin E$, $c(u, v) = 0$
- Assume that every vertex in V lies on some path from the *source vertex* $s \in V$ to the *sink vertex* $t \in V$



Flows

- A *flow* in graph G is a function $f : V \times V \to \mathbb{R}$ that satisfies:
  1. **Capacity constraint:** For all $u, v \in V$, $f(u,v) \leq c(u,v)$ (flow should not exceed capacity)
  2. **Skew symmetry:** For all $u, v \in V$, $f(u,v) = -f(v,u)$ (for convenience; flow defined for all pairs of vertices)
  3. **Flow conservation:** For all $u \in V \setminus \{s,t\}$,

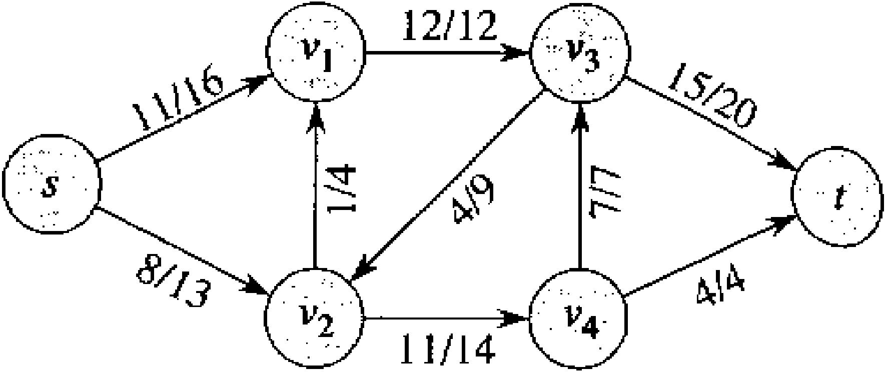
$$\sum_{v \in V} f(u,v) = 0$$

  (flow entering a vertex $=$ flow leaving)
- The *value* of a flow is the flow out of s ($=$ flow into t):

$$|f| = \sum_{v \in V} f(s,v) = \sum_{v \in V} f(v,t)$$

- *Maximum flow problem:* given graph and capacities, find a flow of maximum value

Flow Example



What is the value of this flow?

More Notation

- For convenience, we will also use set notation in f: For $X, Y \subseteq V$,

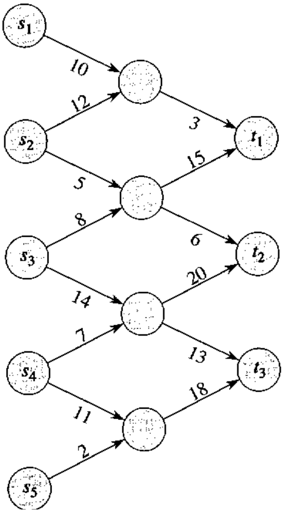
$$f(X,Y) = \sum_{x \in X} \sum_{y \in Y} f(x,y)$$

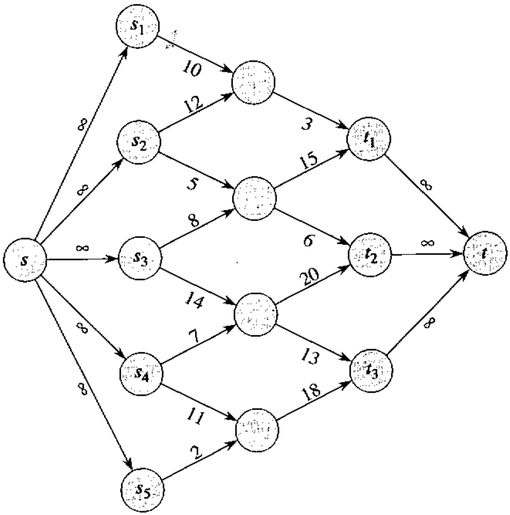
- **Lemma**: If $G = (V, E)$ is a flow network and f is a flow in G, then
  1. For all $X \subseteq V$, $f(X,X) = 0$
  2. For all $X, Y \subseteq V$, $f(X,Y) = -f(Y,X)$
  3. For all $X, Y, Z \subseteq V$ with $X \cap Y = \emptyset$, $f(X \cup Y, Z) = f(X,Z) + f(Y,Z)$ and $f(Z, X \cup Y) = f(Z,X) + f(Z,Y)$

Multiple Sources and Sinks

- Might have cases where there are multiple sources and/or sinks; e.g. if there are multiple factories producing products and/or multiple warehouses to ship to
- Can easily accommodate graphs with multiple sources $s_1, \ldots, s_k$ and multiple sinks $t_1, \ldots, t_\ell$
- Add to G a *supersource* s with an edge (s, s_i) for $i \in \{1, \ldots, k\}$ and a *supersink* t with an edge (t_j, t) for $j \in \{1, \ldots, \ell\}$
- Each new edge has a capacity of ∞

Multiple Sources and Sinks (2)





Ford-Fulkerson Method

- A *method* (rather than specific algorithm) for solving max flow
- Multiple ways of implementing, with varying running times
- Core concepts:
  1. *Residual network:* A network G_f, which is G with capacities reduced based on the amount of flow f already going through it
  2. *Augmenting path:* A simple path from s to t in residual network G_f
     - $\Rightarrow$ If such a path exists, then can push more flow through network
  3. *Cut:* A partition of V into S and T where $s \in S$ and $t \in T$; can measure *net flow* and *capacity* crossing a cut
- Method repeatedly finds an augmenting path in residual network, adds in flow along the path, then updates residual network

Ford-Fulkerson Method (2)

```
1  Initialize flow f to 0
2  while there exists augmenting path p in residual network G_f do
3      augment flow f along p
4  end
5  return f
```

Algorithm 1: Ford-Fulkerson-Method(G, s, t)

Residual Networks

- Given flow network G with capacities c and flow f, *residual network* G_f consists of edges with capacities showing how one can change flow in G
- Define *residual capacity* of an edge as

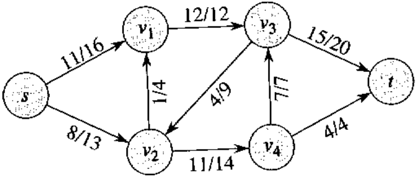
$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \\ 0 & \text{otherwise} \end{cases}$$

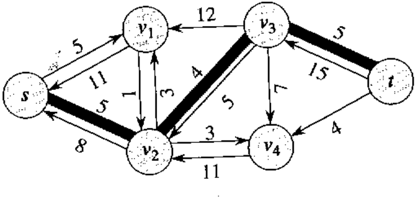
- E.g. if $c(u,v) = 16$ and $f(u,v) = 11$, then $c_f(u,v) = 5$ and $c_f(v,u) = 11$
- Then can define $G_f = (V, E_f)$ as

$$E_f = \{(u,v) \in V \times V : c_f(u,v) > 0\}$$

- So G_f will have some edges not in G, and vice-versa

Residual Networks (2)





Flow Augmentation

- G_f is like a flow network (except that it can have an edge and its reversal); so we can find a flow within it
- If f is a flow in G and f' is a flow in G_f, can define the *augmentation* of f by f' as

$$(f \uparrow f')(u,v) = \begin{cases} f(u,v) + f'(u,v) - f'(v,u) & \text{if } (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$$

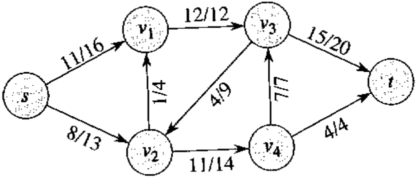
- **Lemma:** $f \uparrow f'$ is a flow in G with value $|f \uparrow f'| = |f| + |f'|$
- **Proof:** Not difficult to show that $f \uparrow f'$ satisfies capacity constraint and and flow conservation; then show that $|f \uparrow f'| = |f| + |f'|$ (pp. 718–719)
- Result: If we can find a flow f' in G_f, we can increase flow in G

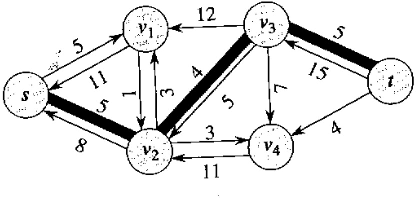
Augmenting Path

- By definition of residual network, an edge $(u, v) \in E_f$ with $c_f(u, v) > 0$ can handle additional flow
- Since edges in E_f all have positive residual capacity, it follows that if there is a simple path p from s to t in G_f, then we can increase flow along each edge in p, thus increasing total flow
- We call p an *augmenting path*
- The amount of flow we can put on p is p's residual capacity:

$$c_f(p) = \min\{c_f(u, v) : (u, v) \text{ is on } p\}$$

Augmenting Path (2)





p is shaded; what is $c_f(p)$?

Augmenting Path (3)

- **Lemma:** Let $G = (V, E)$ be a flow network, f be a flow in G, and p be an augmenting path in G_f. Define $f_p : V \times V \to \mathbb{R}$ as

$$f_p(u,v) = \begin{cases} c_f(p) & \text{if } (u,v) \in p \\ 0 & \text{otherwise} \end{cases}$$

  Then f_p is a flow in G_f with value $|f_p| = c_f(p) > 0$
- **Corollary:** Let G, f, p, and f_p be as above. Then $f \uparrow f_p$ is a flow in G with value $|f \uparrow f_p| = |f| + |f_p| > |f|$
- Thus, every augmenting path increases flow in G
- When do we stop? Will we have a maximum flow if there is no augmenting path?

- Used to prove that once we run out of augmenting paths, we have a maximum flow
- A *cut* (S, T) of a flow network $G = (V, E)$ is a partition of V into $S \subseteq V$ and $T = V \setminus S$ such that $s \in S$ and $t \in T$
- *Net flow* across the cut (S, T) is

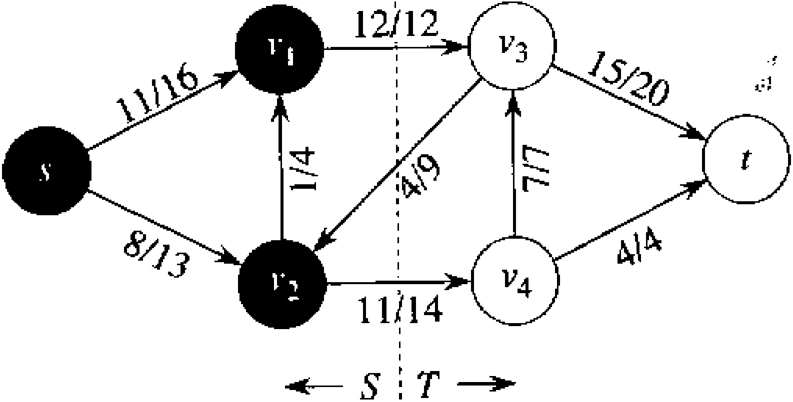
$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u)$$

- *Capacity* of cut (S, T) is

$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$$

- A *minimum cut* is one whose capacity is smallest over all cuts

Max-Flow Min-Cut Theorem (2)



What are $f(S, T)$ and $c(S, T)$?

Max-Flow Min-Cut Theorem (3)

- **Lemma:** For any flow f, the value of f is the same as the net flow across any cut; i.e. $f(S,T) = |f|$ for all cuts (S,T)
- **Corollary:** The value of any flow f in G is upperbounded by the capacity of **any** cut of G
- **Proof:**

$$\begin{aligned} |f| &= f(S,T) \\ &= \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u) \\ &\leq \sum_{u \in S} \sum_{v \in T} f(u,v) \\ &\leq \sum_{u \in S} \sum_{v \in T} c(u,v) \\ &= c(S,T) \end{aligned}$$

Max-Flow Min-Cut Theorem (4)

- **Max-Flow Min-Cut Theorem:** If f is a flow in flow network G, then these statements are equivalent:
  1. f is a maximum flow in G
  2. G_f has no augmenting paths
  3. $|f| = c(S, T)$ for some (i.e. minimum) cut (S, T) of G
- **Proof:** Show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1)
- **(1) $\Rightarrow$ (2):** If G_f has augmenting path p, then $f_p > 0$ and $|f \uparrow f_p| = |f| + |f_p| > |f|$

Max-Flow Min-Cut Theorem (5)

- **(2) $\Rightarrow$ (3):** Assume G_f has no path from s to t and define $S = \{v \in V : s \rightsquigarrow v \text{ in } G_f\}$ and $T = V \setminus S$
  - (S,T) is a cut since it partitions V, $s \in S$ and $t \in T$
  - Consider $u \in S$ and $v \in T$:
    - If $(u,v) \in E$, then $f(u,v) = c(u,v)$ since otherwise $c_f(u,v) > 0 \Rightarrow (u,v) \in E_f \Rightarrow v \in S$
    - If $(v,u) \in E$, then $f(v,u) = 0$ since otherwise we'd have $c_f(u,v) = f(v,u) > 0 \Rightarrow (u,v) \in E_f \Rightarrow v \in S$
    - If $(u,v) \notin E$ and $(v,u) \notin E$, then $f(u,v) = f(v,u) = 0$
  - Thus (by applying the Lemma as well)

$$
\begin{aligned}
|f| = f(S,T) &= \sum_{u \in S}\sum_{v \in T} f(u,v) - \sum_{v \in T}\sum_{u \in S} f(v,u) \\
&= \sum_{u \in S}\sum_{v \in T} c(u,v) - \sum_{v \in T}\sum_{u \in S} 0 = c(S,T)
\end{aligned}
$$

Max-Flow Min-Cut Theorem (6)

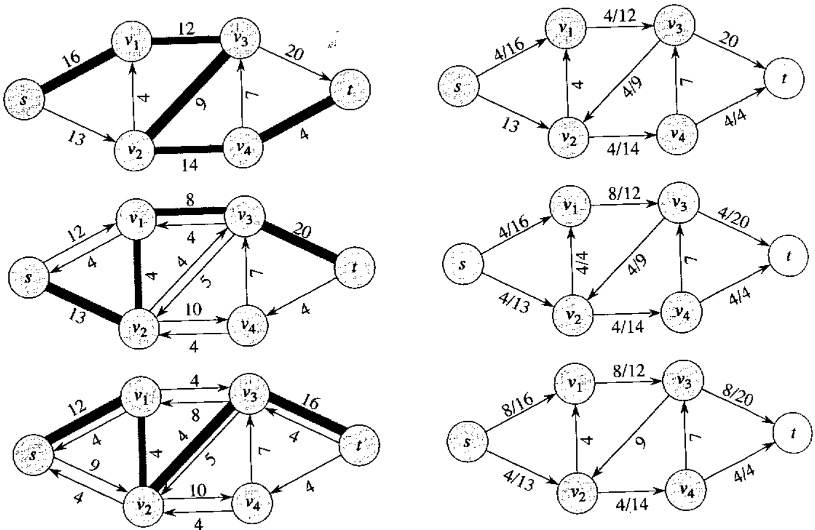
- **(3) $\Rightarrow$ (1):**
  - Corollary says that $|f| \leq c(S', T')$ for all cuts (S', T')
  - We've established that $|f| = c(S, T)$
    - $\Rightarrow$ $|f|$ can't be any larger
    - $\Rightarrow$ f is a maximum flow

Basic Ford-Fulkerson Algorithm

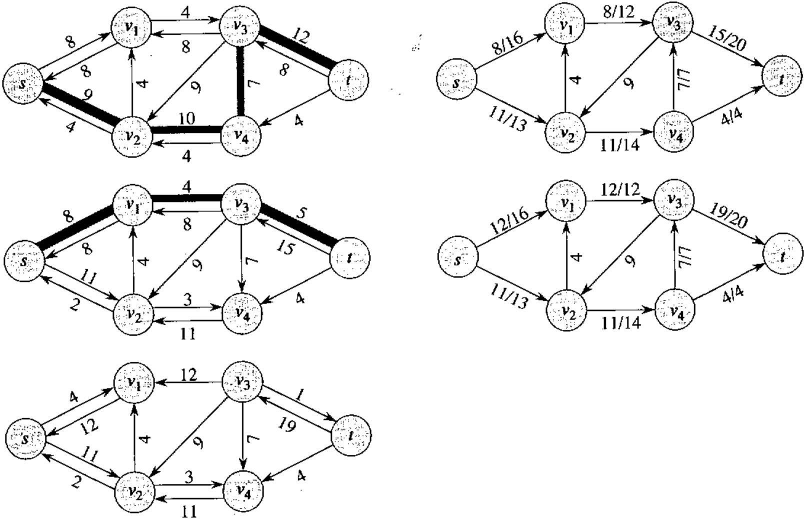
for *each edge* $(u, v) \in E$ **do**
    $f(u, v) = 0$
end
while *there exists path* p *from* s *to* t *in* G_f **do**
    $c_f(p) = \min\{c_f(u, v) : (u, v)$ is in $p\}$
    for *each edge* $(u, v) \in p$ **do**
        if $(u, v) \in E$ **then**
            $f(u, v) = f(u, v) + c_f(p)$
        end
        else
            $f(v, u) = f(v, u) - c_f(p)$
        end
    end
end

Algorithm 2: Ford-Fulkerson(G, s, t)

Ford-Fulkerson Example



Ford-Fulkerson Example (2)



Analysis of Ford-Fulkerson

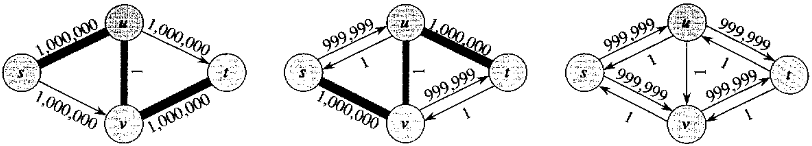
- Assume all of G's capacities are integers
  - If not, but values still rational, can scale them
  - If values irrational, might not converge $\frown$
- If we choose augmenting path arbitrarily, then $|f|$ increases by at least one unit per iteration $\Rightarrow$ number of iterations is $\leq |f^*| =$ value of max flow
- $|E_f| \leq 2|E|$
- Every vertex is on a path from s to t $\Rightarrow |V| = O(|E|)$

$\Rightarrow$ Finding augmenting path via BFS or DFS takes time $O(|E|)$, as do initialization and each augmentation step

- Total time complexity: $O(|E||f^*|)$
- Not polynomial in size of input! (What is size of input?)

Example of Large $|f^*|$

Arbitrary choice of augmenting path can result in small increase in $|f|$ each step



Takes 2×10^6 augmentations

Edmonds-Karp Algorithm

- Uses Ford-Fulkerson Method
- Rather than arbitrary choice of augmenting path p from s to t in G_f, choose one that is shortest in terms of number of edges
  - How can we easily do this?
- Will show time complexity of $O(|V||E|^2)$, independent of $|f^*|$
- Proof based on $\delta_f(u, v)$, which is length of shortest path from u to v in G_f, in terms of number of edges
- **Lemma:** When running Edmonds-Karp on G, for all vertices $v \in V \setminus \{s, t\}$, shortest path distance $\delta_f(u, v)$ in G_f increases monotonically with each flow augmentation

Edmonds-Karp Algorithm (2)

- **Theorem:** When running Edmonds-Karp on G, the total number of flow augmentations is $O(|V||E|)$
- **Proof:** Call an edge (u, v) *critical* on augmenting path p if $c_f(p) = c_f(u, v)$
- When (u, v) is critical for the first time, $\delta_f(s, v) = \delta_f(s, u) + 1$
- At the same time, (u, v) disappears from residual network and does not reappear until its flow decreases, which only happens when (v, u) appears on an augmenting path, at which time

$$\begin{aligned} \delta_{f'}(s, u) &= \delta_{f'}(s, v) + 1 \\ &\geq \delta_f(s, v) + 1 \text{ (from Lemma)} \\ &= \delta_f(s, u) + 2 \end{aligned}$$

- Thus, from the time (u, v) becomes critical to the next time it does, u's distance from s increases by at least 2

Edmonds-Karp Algorithm (3)

- Since u's distance from s is at most $|V|-2$ (because $u \neq t$) and at least 0, edge (u,v) can be critical at most $|V|/2$ times
- There are at most $2|E|$ edges that can be critical in a residual network
- Every augmentation step has at least one critical edge

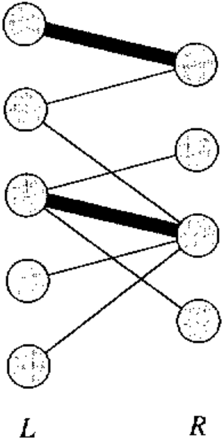
$\Rightarrow$ Number of augmentation steps is $O(|V||E|)$, instead of $O(|f^*|)$ in previous algorithm

$\Rightarrow$ Edmonds-Karp time complexity is $O(|V||E|^2)$

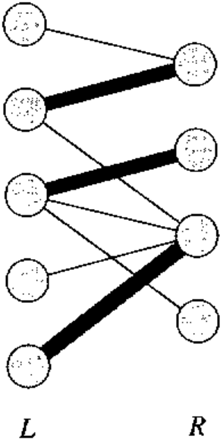
Maximum Bipartite Matching

- In an undirected graph $G = (V, E)$, a *matching* is a subset of edges $M \subseteq E$ such that for all $v \in V$, at most one edge from M is incident on v
- If an edge from M is incident on v, v is *matched*, otherwise *unmatched*
- **Problem:** Find a matching of maximum cardinality
- **Special case:** G is *bipartite*, meaning V partitioned into disjoint sets L and R and all edges of E go between L and R
- Applications: Matching machines to tasks, arranging marriages between interested parties, etc.

Bipartite Matching Example



$|M| = 2$



$|M| = 3$ (maximum)

Casting Bipartite Matching as Max Flow

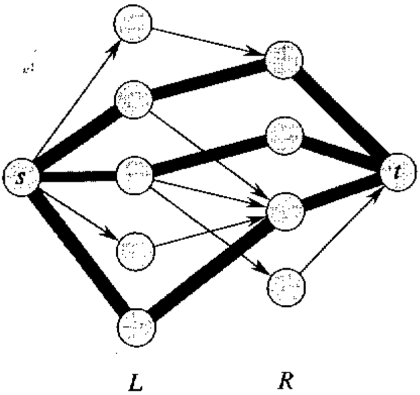
- Can cast bipartite matching problem as max flow
- Given bipartite graph $G = (V, E)$, define *corresponding flow network* $G' = (V', E')$:

$$V' = V \cup \{s, t\}$$

$$E' = \{(s, u) : u \in L\} \cup \{(u, v) : (u, v) \in E\} \cup \{(v, t) : v \in R\}$$

- $c(u, v) = 1$ for all $(u, v) \in E'$

Casting Bipartite Matching as Max Flow (2)



Value of flow across cut $(L \cup \{s\}, R \cup \{t\})$ equals $|M|$

Casting Bipartite Matching as Max Flow (3)

- **Lemma:** Let $G = (V, E)$ be a bipartite graph with V paritioned into L and R and let $G' = (V', E')$ be its corresponding flow network. If M is a matching in G, then there is an *integer-valued flow* f in G' with value $|f| = |M|$. Conversely, if there is an integer-valued flow f in G', then there is a matching M in G with cardinality $|M| = |f|$.
- **Proof:** $\Rightarrow$ If $(u, v) \in M$, set $f(s, u) = f(u, v) = f(v, t) = 1$
  - Set flow of all other edges to 0
  - Flow satisfies capacity constraint and flow conservation
  - Flow across cut $(L \cup \{s\}, R \cup \{t\})$ is $|M|$
  - $\Leftarrow$ Let f be integer-valued flow in G', and set

$$M = \{(u, v) : u \in L, v \in R, f(u, v) > 0\}$$

  - Any flow into u must be exactly 1 in and exactly 1 out on one edge
  - Similar argument for $v \in R$, so M is a matching with $|M| = |f|$

Casting Bipartite Matching as Max Flow (4)

- **Theorem:** If all edges in a flow network have integral capacities, then the Ford-Fulkerson method returns a flow with value that is an integer, and for all $(u, v) \in V$, $f(u, v)$ is an integer
- Since the corresponding flow network for bipartite matching uses all integer capacities, can use Ford-Fulkerson to solve matching problem
- Any matching has cardinality $O(|V|)$, so the corresponding flow network has a maximum flow with value $|f^*| = O(|V|)$, so time complexity of matching is $O(|V||E|)$