

Computer Science & Engineering 423/823

Design and Analysis of Algorithms

Lecture 07 — Single-Source Shortest Paths (Chapter 24)

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Introduction

- ▶ Given a weighted, directed graph $G = (V, E)$ with weight function $w : E \rightarrow \mathbb{R}$
- ▶ The **weight** of path $p = \langle v_0, v_1, \dots, v_k \rangle$ is the sum of the weights of its edges:

$$w(p) = \sum_{i=1}^k w(v_{i-1}, v_i)$$

- ▶ Then the **shortest-path weight** from u to v is

$$\delta(u, v) = \begin{cases} \min\{w(p) : u \overset{p}{\rightsquigarrow} v\} & \text{if there is a path from } u \text{ to } v \\ \infty & \text{otherwise} \end{cases}$$

- ▶ A **shortest path** from u to v is any path p with weight $w(p) = \delta(u, v)$
- ▶ **Applications:** Network routing, driving directions

Types of Shortest Path Problems

Given G as described earlier,

- ▶ **Single-Source Shortest Paths:** Find shortest paths from **source** node s to every other node
- ▶ **Single-Destination Shortest Paths:** Find shortest paths from every node to **destination** t
 - ▶ Can solve with SSSP solution. How?
- ▶ **Single-Pair Shortest Path:** Find shortest path from specific node u to specific node v
 - ▶ Can solve via SSSP; no asymptotically faster algorithm known
- ▶ **All-Pairs Shortest Paths:** Find shortest paths between every pair of nodes
 - ▶ Can solve via repeated application of SSSP, but can do better

Optimal Substructure of a Shortest Path

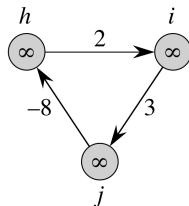
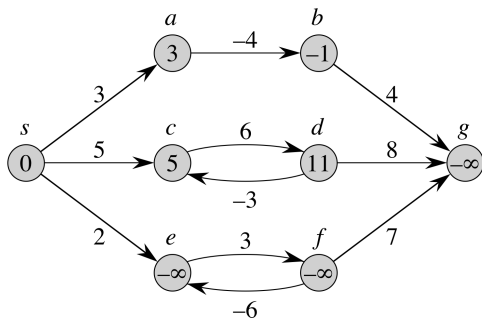
The shortest paths problem has the **optimal substructure property**: If $p = \langle v_0, v_1, \dots, v_k \rangle$ is a SP from v_0 to v_k , then for $0 \leq i \leq j \leq k$, $p_{ij} = \langle v_i, v_{i+1}, \dots, v_j \rangle$ is a SP from v_i to v_j

Proof: Let $p = v_0 \xrightarrow{p_{0i}} v_i \xrightarrow{p_{ij}} v_j \xrightarrow{p_{jk}} v_k$ with weight $w(p) = w(p_{0i}) + w(p_{ij}) + w(p_{jk})$. If there exists a path p'_{ij} from v_i to v_j with $w(p'_{ij}) < w(p_{ij})$, then p is not a SP since $v_0 \xrightarrow{p_{0i}} v_i \xrightarrow{p'_{ij}} v_j \xrightarrow{p_{jk}} v_k$ has less weight than p 

Negative-Weight Edges (1)

- ▶ What happens if the graph G has edges with negative weights?
- ▶ Dijkstra's algorithm cannot handle this, Bellman-Ford can, under the right circumstances (which circumstances?)

Negative-Weight Edges (2)



Cycles

- ▶ What kinds of cycles might appear in a shortest path?
 - ▶ Negative-weight cycle
 - ▶ Zero-weight cycle
 - ▶ Positive-weight cycle

Relaxation

- ▶ Given weighted graph $G = (V, E)$ with source node $s \in V$ and other node $v \in V$ ($v \neq s$), we'll maintain $d[v]$, which is upper bound on $\delta(s, v)$
- ▶ **Relaxation** of an edge (u, v) is the process of testing whether we can decrease $d[v]$, yielding a tighter upper bound

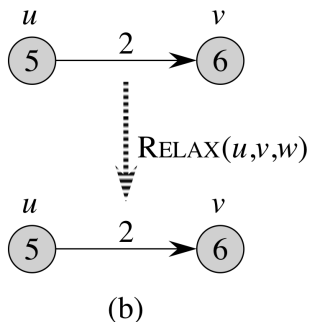
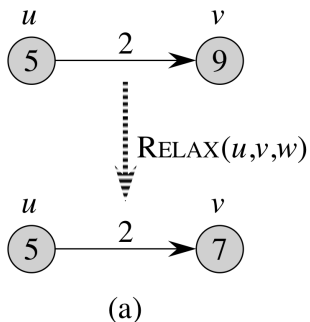
Initialize-Single-Source(G, s)

```
1 for each vertex  $v \in V$  do  
2   |    $d[v] = \infty$  ;  
3   |    $\pi[v] = \text{NIL}$  ;  
4 end  
5  $d[s] = 0$  ;
```

Relax(u, v, w)

```
1 if  $d[v] > d[u] + w(u, v)$  then  
2    $d[v] = d[u] + w(u, v)$  ;  
3    $\pi[v] = u$  ;
```

Relaxation Example



Numbers in nodes are values of d

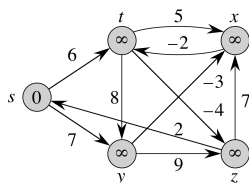
Bellman-Ford Algorithm

- ▶ Works with negative-weight edges and detects if there is a negative-weight cycle
- ▶ Makes $|V| - 1$ passes over all edges, relaxing each edge during each pass
 - ▶ No cycles implies all shortest paths have $\leq |V| - 1$ edges, so that number of relaxations is sufficient

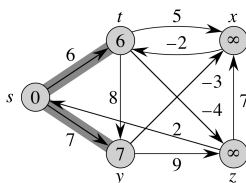
Bellman-Ford(G, w, s)

```
1 INITIALIZE-SINGLE-SOURCE( $G, s$ ) ;
2 for  $i = 1$  to  $|V| - 1$  do
3   | for each edge  $(u, v) \in E$  do
4   |   | RELAX( $u, v, w$ ) ;
5   | end
6 end
7 for each edge  $(u, v) \in E$  do
8   | if  $d[v] > d[u] + w(u, v)$  then
9   |   | return FALSE //  $G$  has a negative-wt cycle ;
10 end
11 return TRUE //  $G$  has no neg-wt cycle reachable frm  $s$  ;
```

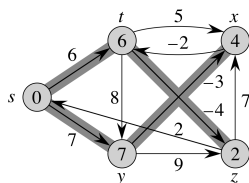
Bellman-Ford Algorithm Example (1)



(a)



(b)

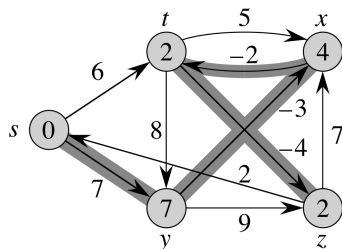


(c)

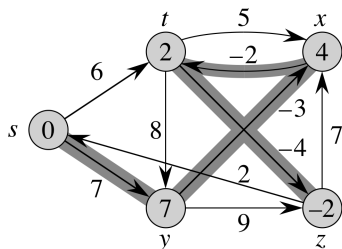
Within each pass, edges relaxed in this order:

$(t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)$

Bellman-Ford Algorithm Example (2)



(d)



(e)

Within each pass, edges relaxed in this order:

(t, x) , (t, y) , (t, z) , (x, t) , (y, x) , (y, z) , (z, x) , (z, s) , (s, t) , (s, y)

Time Complexity of Bellman-Ford Algorithm

- ▶ INITIALIZE-SINGLE-SOURCE takes how much time?
- ▶ RELAX takes how much time?
- ▶ What is time complexity of relaxation steps (nested loops)?
- ▶ What is time complexity of steps to check for negative-weight cycles?
- ▶ What is total time complexity?

Correctness of Bellman-Ford: Finds SP Lengths

- ▶ Assume no negative-weight cycles
- ▶ Since no cycles appear in SPs, every SP has at most $|V| - 1$ edges
- ▶ Then define sets $S_0, S_1, \dots, S_{|V|-1}$:

$$S_k = \{v \in V : \exists s \overset{p}{\rightsquigarrow} v \text{ s.t. } \delta(s, v) = w(p) \text{ and } |p| \leq k\}$$

- ▶ **Loop invariant:** After i th iteration of outer relaxation loop (Line 2), for all $v \in S_i$, we have $d[v] = \delta(s, v)$
 - ▶ aka **path-relaxation property** (Lemma 24.15)
 - ▶ Can prove via induction on i :
 - ▶ Obvious for $i = 0$
 - ▶ If holds for $v \in S_{i-1}$, then definition of relaxation and optimal substructure $\Rightarrow$ holds for $v \in S_i$
- ▶ Implies that, after $|V| - 1$ iterations, $d[v] = \delta(s, v)$ for all $v \in V = S_{|V|-1}$

Correctness of Bellman-Ford: Detects Negative-Weight Cycles

- ▶ Let $c = \langle v_0, v_1, \dots, v_k = v_0 \rangle$ be neg-weight cycle reachable from s :

$$\sum_{i=1}^k w(v_{i-1}, v_i) < 0$$

- ▶ If algorithm incorrectly returns TRUE, then (due to Line 8) for all nodes in the cycle ($i = 1, 2, \dots, k$),

$$d[v_i] \leq d[v_{i-1}] + w(v_{i-1}, v_i)$$

- ▶ By summing, we get

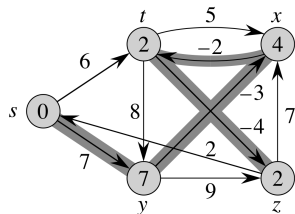
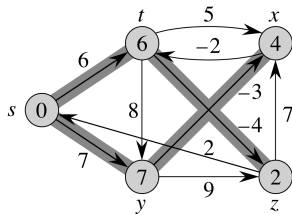
$$\sum_{i=1}^k d[v_i] \leq \sum_{i=1}^k d[v_{i-1}] + \sum_{i=1}^k w(v_{i-1}, v_i)$$

- ▶ Since $v_0 = v_k$, $\sum_{i=1}^k d[v_i] = \sum_{i=1}^k d[v_{i-1}]$
- ▶ This implies that $0 \leq \sum_{i=1}^k w(v_{i-1}, v_i)$, a contradiction



SSSPs in Directed Acyclic Graphs

- ▶ Why did Bellman-Ford have to run $|V| - 1$ iterations of edge relaxations?
- ▶ To confirm that SP information fully propagated to all nodes (path-relaxation property)

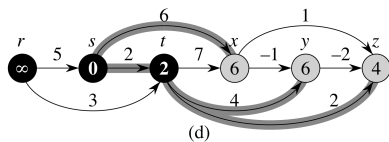
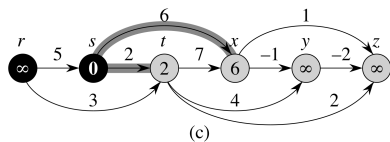
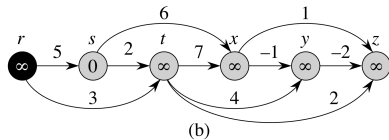
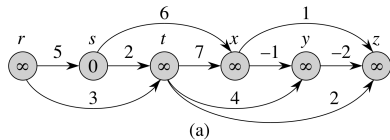


- ▶ What if we knew that, after we relaxed an edge just once, we would be completely done with it?
- ▶ Can do this if G a dag and we relax edges in correct order (what order?)

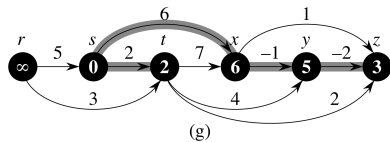
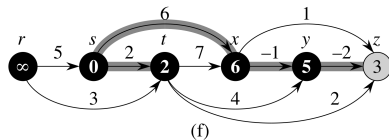
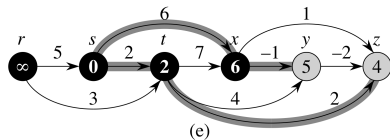
Dag-Shortest-Paths(G, w, s)

```
1 topologically sort the vertices of  $G$  ;  
2 INITIALIZE-SINGLE-SOURCE( $G, s$ ) ;  
3 for each vertex  $u \in V$ , taken in topo sorted  
  order do  
4   | for each  $v \in Adj[u]$  do  
5   |   RELAX( $u, v, w$ ) ;  
6   | end  
7 end
```

SSSP dag Example (1)



SSSP dag Example (2)



Analysis

- ▶ Correctness follows from path-relaxation property similar to Bellman-Ford, except that relaxing edges in topologically sorted order implies we relax the edges of a shortest path in order
- ▶ Topological sort takes how much time?
- ▶ INITIALIZE-SINGLE-SOURCE takes how much time?
- ▶ How many calls to RELAX?
- ▶ What is total time complexity?



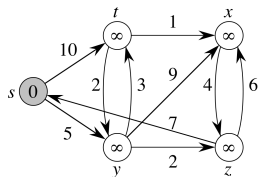
Dijkstra's Algorithm

- ▶ Greedy algorithm
- ▶ Faster than Bellman-Ford
- ▶ Requires all edge weights to be nonnegative
- ▶ Maintains set S of vertices whose final shortest path weights from s have been determined
 - ▶ Repeatedly select $u \in V \setminus S$ with minimum SP estimate, add u to S , and relax all edges leaving u
- ▶ Uses min-priority queue to repeatedly make greedy choice

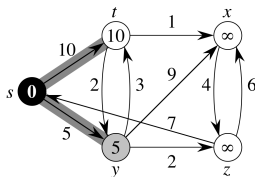
Dijkstra(G, w, s)

```
1 INITIALIZE-SINGLE-SOURCE( $G, s$ ) ;  
2  $S = \emptyset$  ;  
3  $Q = V$  ;  
4 while  $Q \neq \emptyset$  do  
5    $u = \text{EXTRACT-MIN}(Q)$  ;  
6    $S = S \cup \{u\}$  ;  
7   for each  $v \in \text{Adj}[u]$  do  
8      $\text{RELAX}(u, v, w)$  ;  
9   end  
10 end
```

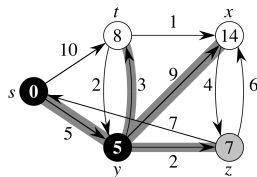
Dijkstra's Algorithm Example (1)



(a)

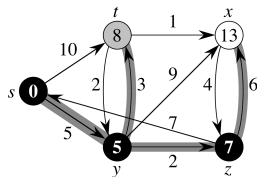


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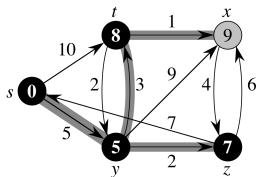


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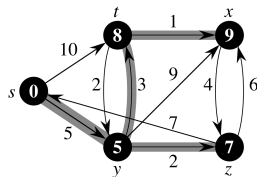
Dijkstra's Algorithm Example (2)



(d)



(e)



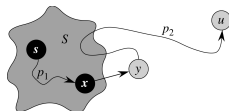
(f)

Time Complexity of Dijkstra's Algorithm

- ▶ Using array to implement priority queue,
 - ▶ INITIALIZE-SINGLE-SOURCE takes how much time?
 - ▶ What is time complexity to create Q ?
 - ▶ How many calls to EXTRACT-MIN?
 - ▶ What is time complexity of EXTRACT-MIN?
 - ▶ How many calls to RELAX?
 - ▶ What is time complexity of RELAX?
 - ▶ What is total time complexity?
- ▶ Using heap to implement priority queue, what are the answers to the above questions?
- ▶ When might you choose one queue implementation over another?

Correctness of Dijkstra's Algorithm

- ▶ **Invariant:** At the start of each iteration of the while loop, $d[v] = \delta(s, v)$ for all $v \in S$
 - ▶ **Proof:** Let u be first node added to S where $d[u] \neq \delta(s, u)$
 - ▶ Let $p = s \xrightarrow{p_1} x \rightarrow y \xrightarrow{p_2} u$ be SP to u and y first node on p in $V - S$
 - ▶ Since y 's predecessor $x \in S$, $d[y] = \delta(s, y)$ due to relaxation of (x, y)
 - ▶ Since y precedes u in p and edge wts non-negative:
 $d[y] = \delta(s, y) \leq \delta(s, u) \leq d[u]$
 - ▶ Since u was chosen before y in line 5, $d[u] \leq d[y]$, so
 $d[y] = \delta(s, y) = \delta(s, u) = d[u]$, a contradiction



Since all vertices eventually end up in S , get correctness of the algorithm



Linear Programming

- ▶ Given an $m \times n$ matrix A and a size- m vector b and a size- n vector c , find a vector x of n elements that maximizes $\sum_{i=1}^n c_i x_i$ subject to $Ax \leq b$

- ▶ E.g., $c = \begin{bmatrix} 2 & -3 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ -1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 22 \\ 4 \\ -8 \end{bmatrix}$

implies:

maximize $2x_1 - 3x_2$ **subject to**

$$x_1 + x_2 \leq 22$$

$$x_1 - 2x_2 \leq 4$$

$$x_1 \geq 8$$

- ▶ **Solution:** $x_1 = 16$, $x_2 = 6$

Difference Constraints and Feasibility

- ▶ **Decision version of this problem:** No objective function to maximize; simply want to know if there exists a **feasible solution**, i.e., an x that satisfies $Ax \leq b$
- ▶ Special case is when each row of A has exactly one 1 and one -1 , resulting in a set of **difference constraints** of the form

$$x_j - x_i \leq b_k$$

- ▶ **Applications:** Any application in which a certain amount of time must pass between events (x variables represent times of events)

Difference Constraints and Feasibility (2)

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \quad \text{and } b = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 5 \\ 4 \\ -1 \\ -3 \\ -3 \end{bmatrix}$$

Difference Constraints and Feasibility (3)

Is there a setting for $x_1, \dots, x_5$ satisfying:

$$x_1 - x_2 \leq 0$$

$$x_1 - x_5 \leq -1$$

$$x_2 - x_5 \leq 1$$

$$x_3 - x_1 \leq 5$$

$$x_4 - x_1 \leq 4$$

$$x_4 - x_3 \leq -1$$

$$x_5 - x_3 \leq -3$$

$$x_5 - x_4 \leq -3$$

One solution: $x = (-5, -3, 0, -1, -4)$

Constraint Graphs

- ▶ Can represent instances of this problem in a **constraint graph** $G = (V, E)$
- ▶ Define a vertex for each variable, plus one more: If variables are $x_1, \dots, x_n$, get $V = \{v_0, v_1, \dots, v_n\}$
- ▶ Add a directed edge for each constraint, plus an edge from v_0 to each other vertex:

$$E = \{(v_i, v_j) : x_j - x_i \leq b_k \text{ is a constraint}\} \\ \cup \{(v_0, v_1), (v_0, v_2), \dots, (v_0, v_n)\}$$

- ▶ Weight of edge (v_i, v_j) is b_k , weight of (v_0, v_ℓ) is 0 for all $\ell \neq 0$

Constraint Graph Example

$$x_1 - x_2 \leq 0$$

$$x_1 - x_5 \leq -1$$

$$x_2 - x_5 \leq 1$$

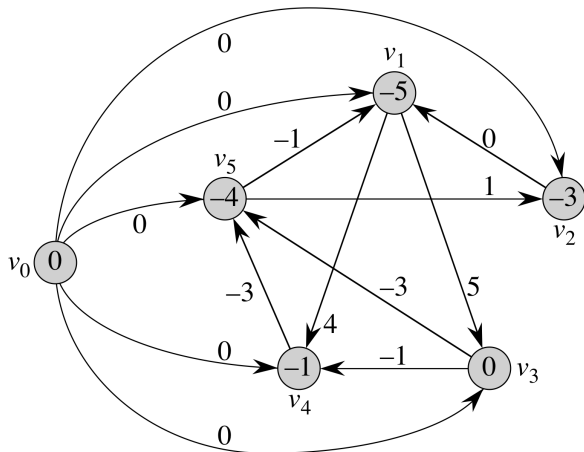
$$x_3 - x_1 \leq 5$$

$$x_4 - x_1 \leq 4$$

$$x_4 - x_3 \leq -1$$

$$x_5 - x_3 \leq -3$$

$$x_5 - x_4 \leq -3$$



$(-5, -3, 0, -1, -4)$

Solving Feasibility with Bellman-Ford

Theorem: Let G be constraint graph for system of difference constraints. If G has a negative-weight cycle, then there is no feasible solution. If G has no negative-weight cycle, then a feasible solution is

$$x = [\delta(v_0, v_1), \delta(v_0, v_2), \dots, \delta(v_0, v_n)]$$

- **Proof:** For any edge $(v_i, v_j) \in E$, triangle inequality says

$$\delta(v_0, v_j) \leq \delta(v_0, v_i) + w(v_i, v_j), \text{ so}$$

$$\delta(v_0, v_j) - \delta(v_0, v_i) \leq w(v_i, v_j)$$

⇒ $x_i = \delta(v_0, v_i)$ and $x_j = \delta(v_0, v_j)$ satisfies constraint

$$x_i - x_j \leq w(v_i, v_j)$$

- If there is a negative-weight cycle

$c = \langle v_i, v_{i+1}, \dots, v_k = v_i \rangle$, then there is a system of inequalities $x_{i+1} - x_i \leq w(v_i, v_{i+1})$,

$$x_{i+2} - x_{i+1} \leq w(v_{i+1}, v_{i+2}), \dots, x_k - x_{k-1} \leq w(v_{k-1}, v_k).$$

Summing both sides gives $0 \leq w(c) < 0$, implying that a negative-weight cycle indicates no solution



Can solve with Bellman-Ford in time $O(n^2 + nm)$