Computer Science & Engineering 423/823
Design and Analysis of Algorithms

Lecture 02 — Medians and Order Statistics (Chapter 9)

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Introduction

- Given an array A of n distinct numbers, the ith **order statistic** of A is its ith smallest element
  - $i = 1 \Rightarrow$ minimum
  - $i = n \Rightarrow$ maximum
  - $i = \lfloor (n+1)/2 \rfloor \Rightarrow$ (lower) median
- E.g. if $A = [8, 5, 3, 10, 4, 12, 6]$ then min = 3, max = 12, median = 6, 3rd order stat = 5
- **Problem:** Given array A of n elements and a number $i \in \{1, \ldots, n\}$, find the ith order statistic of A
- There is an obvious solution to this problem. What is it? What is its time complexity?
  - Can we do better? What if we only focus on $i = 1$ or $i = n$?

Minimum(A)

```
small = A[1] ;
for i = 2 to n do
    if small > A[i] then
        small = A[i] ;
end
return small ;
```


Efficiency of Minimum(A)

- Loop is executed $n - 1$ times, each with one comparison
  - $\Rightarrow$ Total $n - 1$ comparisons
- Can we do better? **NO!**
- **Lower Bound:** Any algorithm finding minimum of n elements will need at least $n - 1$ comparisons
  - Proof of this comes from fact that no element of A can be considered for elimination as the minimum until it's been shown to be greater than at least one other element
  - Imagine that all elements still eligible to be smallest are in a bucket, and are removed only after it is shown to be $>$ some other element
  - Since each comparison removes at most one element from the bucket, at least $n - 1$ comparisons are needed to remove all but one from the bucket

Correctness of Minimum(A)

- Observe that the algorithm always maintains the **invariant** that at the end of each loop iteration, *small* holds the minimum of $A[1 \cdots i]$
  - Easily shown by induction
- Correctness follows by observing that $i == n$ before **return** statement

Simultaneous Minimum and Maximum

- Given array A with n elements, find both its minimum and maximum
- What is the obvious algorithm? What is its (non-asymptotic) time complexity?
- Can we do better?

MinAndMax(A, n)

```
1  large = max(A[1], A[2]) ;
2  small = min(A[1], A[2]) ;
3  for i = 2 to ⌊n/2⌋ do
4  |  large = max(large, max(A[2i − 1], A[2i])) ;
5  |  small = min(small, min(A[2i − 1], A[2i])) ;
6  end
7  if n is odd then
8  |  large = max(large, A[n]) ;
9  |  small = min(small, A[n]) ;
10 return (large, small) ;
```


Explanation of MinAndMax

- Idea: For each pair of values examined in the loop, compare them directly
- For each such pair, compare the smaller one to *small* and the larger one to *large*
- Example: $A = [8, 5, 3, 10, 4, 12, 6]$
  - Initialization: $large = 8$, $small = 5$
  - Compare 3 to 10: $large = \max(8, 10) = 10$, $small = \min(5, 3) = 3$
  - Compare 4 to 12: $large = \max(10, 12) = 12$, $small = \min(3, 4) = 3$
  - Final: $large = \max(12, 6) = 12$, $small = \min(3, 6) = 3$

Efficiency of MinAndMax

- How many comparisons does MinAndMax make?
- Initialization on Lines 1 and 2 requires only one comparison
- Each iteration through the loop requires one comparison between $A[2i-1]$ and $A[2i]$ and then one comparison to each of *large* and *small*, for a total of three
- Lines 8 and 9 require one comparison each
- Total is at most $1 + 3(\lfloor n/2 \rfloor - 1) + 2 \le 3\lfloor n/2 \rfloor$, which is better than $2n - 3$ for finding minimum and maximum separately

Selection of the ith Smallest Value

- Now to the general problem: Given A and i, return the ith smallest value in A
- Obvious solution is sort and return ith element
- Time complexity is $\Theta(n \log n)$
- Can we do better?

Selection of the ith Smallest Value (2)

- New algorithm: Divide and conquer strategy
- Idea: Somehow discard a constant fraction of the current array after spending only linear time
  - If we do that, we'll get a better time complexity
  - More on this later
- Which fraction do we discard?

Select(A, p, r, i)

```
if p == r then
|  return A[p] ;
q = Partition(A, p, r) // Like Partition in Quicksort ;
k = q − p + 1 // Size of A[p ··· q] ;
if i == k then
|  return A[q] // Pivot value is the answer ;
else if i < k then
|  return Select(A, p, q − 1, i) // Answer is in left subarray ;
else
|  return Select(A, q + 1, r, i − k) // Answer is in right subarray ;
```

Returns ith smallest element from $A[p \cdots r]$

What is Select Doing?

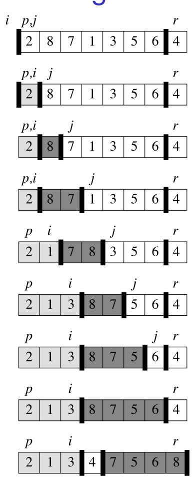
- Like in Quicksort, Select first calls Partition, which chooses a **pivot element** q, then reorders A to put all elements $< A[q]$ to the left of $A[q]$ and all elements $> A[q]$ to the right of $A[q]$
- E.g. if $A = [1, 7, 5, 4, 2, 8, 6, 3]$ and pivot element is 5, then result is $A' = [1, 4, 2, 3, 5, 7, 8, 6]$
- If $A[q]$ is the element we seek, then return it
- If sought element is in left subarray, then recursively search it, and ignore right subarray
- If sought element is in right subarray, then recursively search it, and ignore left subarray

Partition(A, p, r)

```
x = ChoosePivotElement(A, p, r) // Returns index of pivot ;
exchange A[x] with A[r] ;
i = p − 1 ;
for j = p to r − 1 do
    if A[j] ≤ A[r] then
        i = i + 1 ;
        exchange A[i] with A[j] ;
end
exchange A[i + 1] with A[r] ;
return i + 1 ;
```

Chooses a pivot element and partitions $A[p \cdots r]$ around it

Partitioning the Array: Example (Fig 7.1)



Compare each element $A[j]$ to x ($= 4$) and swap with $A[i]$ if $A[j] \leq x$

Choosing a Pivot Element

- Choice of pivot element is critical to low time complexity
- Why?
- What is the best choice of pivot element to partition $A[p \cdots r]$?

Choosing a Pivot Element (2)

- Want to pivot on an element that it as close as possible to being the median
- Of course, we don't know what that is
- Will do **median of medians** approach to select pivot element

Median of Medians

- Given (sub)array A of n elements, partition A into $m = \lfloor n/5 \rfloor$ groups of 5 elements each, and at most one other group with the remaining $n \mod 5$ elements
- Make an array $A' = [x_1, x_2, \ldots, x_{\lceil n/5 \rceil}]$, where x_i is median of group i, found by sorting (in constant time) group i
- Call Select($A', 1, \lceil n/5 \rceil, \lfloor (\lceil n/5 \rceil + 1)/2 \rfloor$)
  - Let value returned be y
  - In linear time, scan $A[p \cdots r]$ and return y's index i
  - Return i as result of ChoosePivotElement(A, p, r)

Example

- Outside of class, get with your team and work this example: Find the 4th smallest element of $A = [4, 9, 12, 17, 6, 5, 21, 14, 8, 11, 13, 29, 3]$
- Show results for each step of Select, Partition, and ChoosePivotElement
- **Good practice for the quiz!**

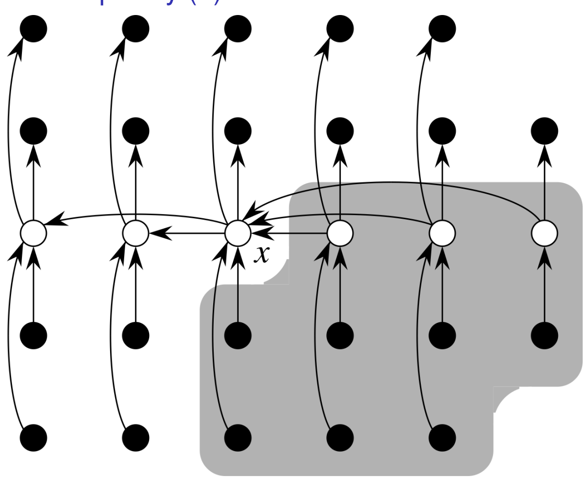
Time Complexity

- Key to time complexity analysis is lower bounding fraction of elements discarded at each recursive call to Select
- On next slide, medians and median (x) of medians are marked, arrows indicate what is guaranteed to be greater than what
- Since x is less than at least half of the other medians (ignoring group with < 5 elements and x's group) and each of those medians is less than 2 elements, we get that the number of elements x is less than is at least

$$3\left(\left\lceil \frac{1}{2}\left\lceil \frac{n}{5} \right\rceil \right\rceil - 2\right) \geq \frac{3n}{10} - 6 \geq n/4 \qquad \text{(if } n \geq 120\text{)}$$

- Similar argument shows that at least $3n/10 - 6 \geq n/4$ elements are less than x
- Thus, if $n \geq 120$, each recursive call to Select is on at most $3n/4$ elements

Time Complexity (2)



Time Complexity (3)

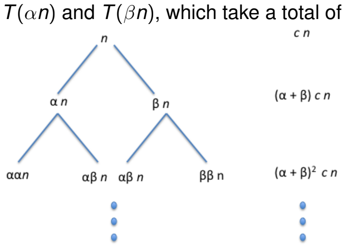
- Develop **recurrence** describing Select's time complexity
- Let $T(n)$ be total time for Select to run on input of size n
- Choosing a pivot element takes time $O(n)$ to split into size-5 groups and time $T(n/5)$ to recursively find the median of medians
- Once pivot element chosen, partitioning n elements takes $O(n)$ time
- Recursive call to Select takes time at most $T(3n/4)$
- Thus we get

$$T(n) \leq T(n/5) + T(3n/4) + O(n)$$

- Can express as $T(\alpha n) + T(\beta n) + O(n)$ for $\alpha = 1/5$ and $\beta = 3/4$
- **Theorem:** For recurrences of the form $T(\alpha n) + T(\beta n) + O(n)$ for $\alpha + \beta < 1$, $T(n) = O(n)$
- Thus Select has time complexity $O(n)$

Proof of Theorem

Top $T(n)$ takes $O(n)$ time ($= cn$ for some constant c). Then calls to $T(\alpha n)$ and $T(\beta n)$, which take a total of $(\alpha + \beta)cn$ time, and so on.



Summing these infinitely yields (since $\alpha + \beta < 1$)

$$cn(1 + (\alpha + \beta) + (\alpha + \beta)^2 + \cdots) = \frac{cn}{1 - (\alpha + \beta)} = c'n = O(n)$$

Master Method

- Another useful tool for analyzing recurrences
- **Theorem:** Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined as $T(n) = aT(n/b) + f(n)$. Then $T(n)$ is bounded as follows.
  1. If $f(n) = O(n^{\log_b a - \epsilon})$ for constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
  2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
  3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for constant $c < 1$ and sufficiently large n, then $T(n) = \Theta(f(n))$
- E.g. for Select, can apply theorem on $T(n) < 2T(3n/4) + O(n)$ (note the slack introduced) with $a = 2$, $b = 4/3$, $\epsilon = 1.4$ and get $T(n) = O\left(n^{\log_{4/3} 2}\right) = O\left(n^{2.41}\right)$

$\Rightarrow$ Not as tight for this recurrence