Homework 5

Assigned November 24, 2019
Due December 8, 2019 on Canvas

CSCE 823 students have to do all problems for full credit. CSCE 423 students need to do only the Core Problems for full credit, and may do the Advanced Problem for bonus points.

For this homework assignment, you are to work in the team that you established in Homework 0. This will be your collaborative team for the rest of the term. You may freely discuss solutions to exercises within your team, and you are to submit a single pdf file from your team. **The internet is not an allowed resource on this homework!**

Clarity of presentation is important. You should give a clear description of all your algorithms, each with a proof of correctness and analysis of time complexity. You must submit your solutions in a single pdf file via Canvas, and are encouraged to prepare your solutions in LaTeX. **Only** pdf will be accepted, and you should submit only one pdf file for Questions 2–4. When you submit for Question 1, submit a second pdf file to the *Questions* assignment in Canvas.

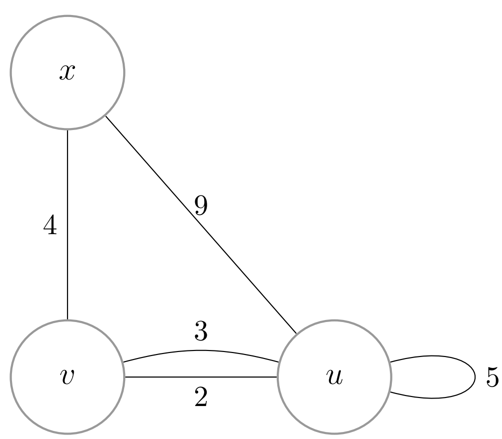
Core Problems

1. **(bonus points, but mandatory submission)** Present one question that you have on the lectures and/or textbook on NP-completeness. This question should be thoughtful and nontrivial, and suggest depth of knowledge in the material. Also, present what you consider to be a reasonable (doesn't have to be completely correct) answer to this problem.
Your question and answer should be submitted to the *Questions* assignment in Canvas in a pdf file separate from the rest of your homework submission.

2. **(30 points)** Prove that the following problem, EXACT-LENGTH-TOUR (ELT), is NP-complete.

> An instance of this problem is $\langle G, w, v, \ell \rangle$, where $G = (V, E)$ is a weighted, undirected graph (where loops and multiple edges between vertices are allowed), w is a positive, integer-valued weight function on the edges E, $v \in V$, and integer $\ell \geq 0$. You must determine if there exists a tour in G that starts at vertex v, never repeats an edge, and returns to v after travelling a distance of exactly ℓ.

For example, for the graph G to the right with weight function w indicated on the edges, instance $\langle G, w, v, 10 \rangle$ would have a "yes" answer, since path $\langle v, u, u, v \rangle$ has total weight $\ell = 10$. (The answer is also "yes" for $\ell = 5$ and some other values, but is "no" for $\ell = 11$ since the weight-3 edge cannot be repeated.)

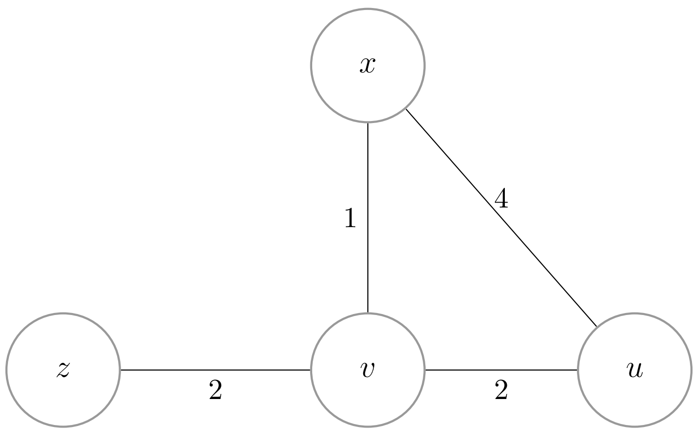


In your proof, reduce from SUBSET-SUM. Notice that you can create any graph you want, and the graph does not have to be simple. You may also reduce from HAM-CYCLE.

3. **(35 points)** Consider the following problem.
You are given as an instance $\langle G, f \rangle$, where $G = (V, E)$ is a simple undirected weighted graph and $f \leq |V|$ is an integer. Your goal is to pick the location for f vertices of G as locations of tokens so that the *length of the shortest path* between *any* vertex $v \in V$ and its closest token is the smallest possible. From this optimization problem we create the following decision problem that we will call the TOKEN-PLACEMENT problem:

> An instance of the problem is $\langle G, f, d \rangle$, where $G = (V, E)$ is an undirected weighted graph, $f \leq |V|$ is an integer, and $d \geq 0$ is real number. The question to answer is: is it possible to select f vertices of G as token locations so that the length of the shortest path between any vertex $v \in V$ and its nearest token is at most d?

For example, for the graph G below with weights indicated on the edges, instance $\langle G, 1, 2 \rangle$ would have a "yes" answer, since placing a token at node v results in every node in V being within distance $d = 2$ of a token. (If $d = 1$ then the answer is "no" since there is no way to place a single token to be within unit distance of every vertex.)



Prove that TOKEN-PLACEMENT is NP-complete.
In your proof, reduce from a problem described in the textbook's chapter on NP-completeness.

Advanced Problem

4. **(40 points)** Prove that the following problem, BALANCEABLE-SET, is NP-complete.

> An instance of this problem is $\langle S \rangle$, where S is a multiset of positive integers (where elements can be repeated). The question is whether S can be split into two *balanced sets* $A \subseteq S$ and $\overline{A} = S \setminus A$ such that
> $$\sum_{x \in A} x = \sum_{x \in \overline{A}} x \ .$$

For example, the multiset $S = \{3, 4, 7, 6, 3, 9, 1, 1\}$ would have a "yes" answer, since it can be balanced between $A = \{3, 4, 6, 3, 1\}$ and $\overline{A} = \{7, 9, 1\}$.
In your proof, reduce from a problem described in the textbook's chapter on NP-completeness.