Computer Science & Engineering 423/823
Design and Analysis of Algorithms
Lecture 05 — Single-Source Shortest Paths (Chapter 24)

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Introduction

- Given a weighted, directed graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$
- The **weight** of path $p = \langle v_0, v_1, \ldots, v_k \rangle$ is the sum of the weights of its edges:
$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$$
- Then the **shortest-path weight** from u to v is
$$\delta(u,v) = \begin{cases} \min\{w(p) : u \overset{p}{\rightsquigarrow} v\} & \text{if there is a path from } u \text{ to } v \\ \infty & \text{otherwise} \end{cases}$$
- A **shortest path** from u to v is any path p with weight $w(p) = \delta(u, v)$
- **Applications:** Network routing, driving directions

Types of Shortest Path Problems

Given G as described earlier,

- **Single-Source Shortest Paths:** Find shortest paths from **source** node s to every other node
- **Single-Destination Shortest Paths:** Find shortest paths from every node to **destination** t
  - Can solve with SSSP solution. How?
- **Single-Pair Shortest Path:** Find shortest path from specific node u to specific node v
  - Can solve via SSSP; no asymptotically faster algorithm known
- **All-Pairs Shortest Paths:** Find shortest paths between every pair of nodes
  - Can solve via repeated application of SSSP, but can do better

Optimal Substructure of a Shortest Path

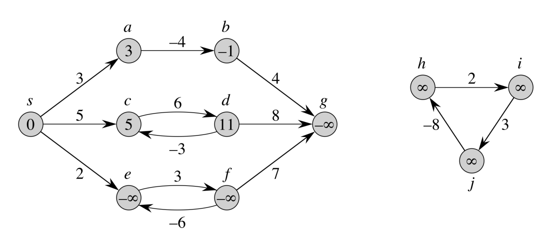
The shortest paths problem has the **optimal substructure property**: If $p = \langle v_0, v_1, \ldots, v_k \rangle$ is a SP from v_0 to v_k, then for $0 \le i \le j \le k$, $p_{ij} = \langle v_i, v_{i+1}, \ldots, v_j \rangle$ is a SP from v_i to v_j

Proof: Let $p = v_0 \overset{p_{0i}}{\rightsquigarrow} v_i \overset{p_{ij}}{\rightsquigarrow} v_j \overset{p_{jk}}{\rightsquigarrow} v_k$ with weight $w(p) = w(p_{0i}) + w(p_{ij}) + w(p_{jk})$. If there exists a path p'_{ij} from v_i to v_j with $w(p'_{ij}) < w(p_{ij})$, then p is not a SP since $v_0 \overset{p_{0i}}{\rightsquigarrow} v_i \overset{p'_{ij}}{\rightsquigarrow} v_j \overset{p_{jk}}{\rightsquigarrow} v_k$ has less weight than p □

Negative-Weight Edges (1)

- What happens if the graph G has edges with negative weights?
- Dijkstra's algorithm cannot handle this, Bellman-Ford can, under the right circumstances (which circumstances?)

Negative-Weight Edges (2)



Cycles

- What kinds of cycles might appear in a shortest path?
  - Negative-weight cycle
  - Zero-weight cycle
  - Positive-weight cycle

Relaxation

- Given weighted graph $G = (V, E)$ with source node $s \in V$ and other node $v \in V$ $(v \neq s)$, we'll maintain $d[v]$, which is upper bound on $\delta(s, v)$
- **Relaxation** of an edge (u, v) is the process of testing whether we can decrease $d[v]$, yielding a tighter upper bound

Initialize-Single-Source(G, s)

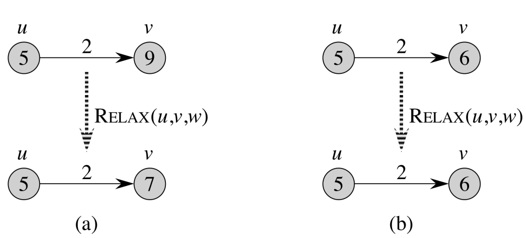
for *each vertex* $v \in V$ **do**
    $d[v] = \infty$
    $\pi[v] = \text{NIL}$
end
$d[s] = 0$

Relax(u, v, w)

if $d[v] > d[u] + w(u, v)$ **then**
    $d[v] = d[u] + w(u, v)$
    $\pi[v] = u$

How do we know that we can tighten $d[v]$ like this?

Relaxation Example



Numbers in nodes are values of d

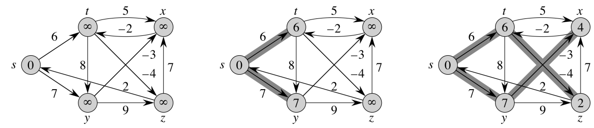
Bellman-Ford Algorithm

- Works with negative-weight edges and detects if there is a negative-weight cycle
- Makes $|V| - 1$ passes over all edges, relaxing each edge during each pass
  - No cycles implies all shortest paths have $\leq |V| - 1$ edges, so that number of relaxations is sufficient

Bellman-Ford(G, w, s)

```
1  Initialize-Single-Source(G, s)
2  for i = 1 to |V| − 1 do
3      for each edge (u, v) ∈ E do
4          Relax(u, v, w)
5      end
6  end
7  for each edge (u, v) ∈ E do
8      if d[v] > d[u] + w(u, v) then
9          return FALSE // G has a negative-wt cycle
10
11 end
12 return TRUE // G has no neg-wt cycle reachable frm s
```

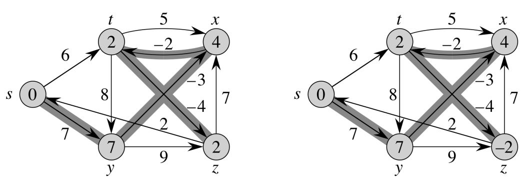
Bellman-Ford Algorithm Example (1)



(a) (b) (c)

Within each pass, edges relaxed in this order:
$(t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)$

Bellman-Ford Algorithm Example (2)



(d) (e)

Within each pass, edges relaxed in this order:
$(t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)$

Time Complexity of Bellman-Ford Algorithm

- Initialize-Single-Source takes how much time?
- Relax takes how much time?
- What is time complexity of relaxation steps (nested loops)?
- What is time complexity of steps to check for negative-weight cycles?
- What is total time complexity?

Correctness of Bellman-Ford: Finds SP Lengths

- Assume no negative-weight cycles
- Since no cycles appear in SPs, every SP has at most $|V| - 1$ edges
- Then define sets $S_0, S_1, \ldots S_{|V|-1}$:

$$S_k = \{v \in V : \exists s \overset{p}{\rightsquigarrow} v \text{ s.t. } \delta(s, v) = w(p) \text{ and } |p| \le k\}$$

- **Loop invariant:** After ith iteration of outer relaxation loop (Line 2), for all $v \in S_i$, we have $d[v] = \delta(s, v)$
  - aka **path-relaxation property** (Lemma 24.15)
  - Can prove via induction on i:
    - Obvious for $i = 0$
    - If holds for $v \in S_{i-1}$, then definition of relaxation and optimal substructure $\Rightarrow$ holds for $v \in S_i$
- Implies that, after $|V| - 1$ iterations, $d[v] = \delta(s, v)$ for all $v \in V = S_{|V|-1}$

Correctness of Bellman-Ford: Detects Negative-Weight Cycles

- Let $c = \langle v_0, v_1, \ldots, v_k = v_0 \rangle$ be neg-weight cycle reachable from s:

$$\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$$

- If algorithm incorrectly returns TRUE, then (due to Line 8) for all nodes in the cycle ($i = 1, 2, \ldots, k$),

$$d[v_i] \le d[v_{i-1}] + w(v_{i-1}, v_i)$$

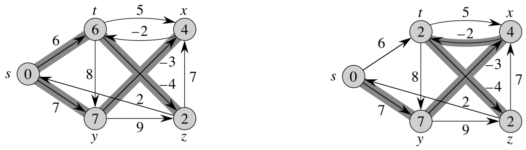
- By summing, we get

$$\sum_{i=1}^{k} d[v_i] \le \sum_{i=1}^{k} d[v_{i-1}] + \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

- Since $v_0 = v_k$, $\sum_{i=1}^{k} d[v_i] = \sum_{i=1}^{k} d[v_{i-1}]$
- This implies that $0 \le \sum_{i=1}^{k} w(v_{i-1}, v_i)$, a contradiction □

SSSPs in Directed Acyclic Graphs

- Why did Bellman-Ford have to run $|V| - 1$ iterations of edge relaxations?
- To confirm that SP information fully propagated to all nodes (path-relaxation property)

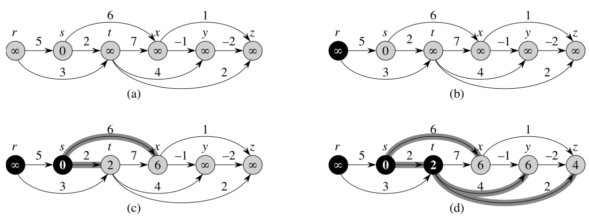


- What if we knew that, after we relaxed an edge just once, we would be completely done with it?
- Can do this if G a dag and we relax edges in correct order (what order?)

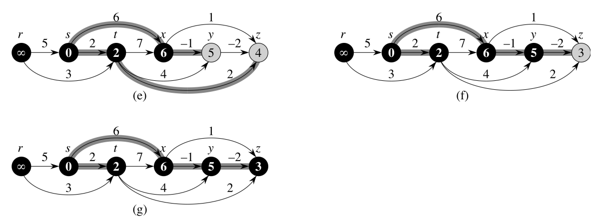
Dag-Shortest-Paths(G, w, s)

```
topologically sort the vertices of G
INITIALIZE-SINGLE-SOURCE(G, s)
for each vertex u ∈ V, taken in topo sorted order do
    for each v ∈ Adj[u] do
        RELAX(u, v, w)
    end
end
```

SSSP dag Example (1)



SSSP dag Example (2)



Analysis

- Correctness follows from path-relaxation property similar to Bellman-Ford, except that relaxing edges in topologically sorted order implies we relax the edges of a shortest path in order □
- Topological sort takes how much time?
- INITIALIZE-SINGLE-SOURCE takes how much time?
- How many calls to RELAX?
- What is total time complexity?

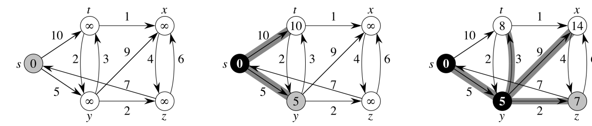
Dijkstra's Algorithm

- Greedy algorithm
- Faster than Bellman-Ford
- Requires all edge weights to be nonnegative
- Maintains set S of vertices whose final shortest path weights from s have been determined
  - Repeatedly select $u \in V \setminus S$ with minimum SP estimate, add u to S, and relax all edges leaving u
- Uses min-priority queue to repeatedly make greedy choice

Dijkstra(G, w, s)

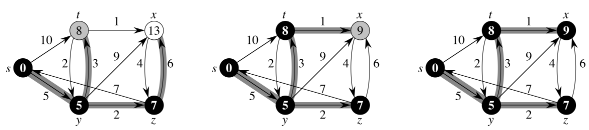
```
1  INITIALIZE-SINGLE-SOURCE(G, s)
2  S = ∅
3  Q = V
4  while Q ≠ ∅ do
5  |   u = EXTRACT-MIN(Q)
6  |   S = S ∪ {u}
7  |   for each v ∈ Adj[u] do
8  |   |   RELAX(u, v, w)
9  |   end
10 end
```

Dijkstra's Algorithm Example (1)



(a) (b) (c)

Dijkstra's Algorithm Example (2)



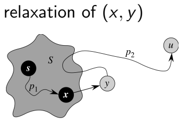
(d) (e) (f)

Time Complexity of Dijkstra's Algorithm

- Using array to implement priority queue,
  - INITIALIZE-SINGLE-SOURCE takes how much time?
  - What is time complexity to create Q?
  - How many calls to EXTRACT-MIN?
  - What is time complexity of EXTRACT-MIN?
  - How many calls to RELAX?
  - What is time complexity of RELAX?
  - What is total time complexity?
- Using heap to implement priority queue, what are the answers to the above questions?
- When might you choose one queue implementation over another?

Correctness of Dijkstra's Algorithm

- **Invariant:** At the start of each iteration of the while loop, $d[v] = \delta(s, v)$ for all $v \in S$
  - **Proof:** Let u be first node added to S where $d[u] \neq \delta(s, u)$
  - Let $p = s \overset{p_1}{\leadsto} x \to y \overset{p_2}{\leadsto} u$ be SP to u and y first node on p in $V - S$
  - Since y's predecessor $x \in S$, $d[y] = \delta(s, y)$ due to relaxation of (x, y)
  - Since y precedes u in p and edge wts non-negative: $d[y] = \delta(s, y) \leq \delta(s, u) \leq d[u]$



  - Since u was chosen before y in line 5, $d[u] \leq d[y]$, so $d[y] = \delta(s, y) = \delta(s, u) = d[u]$, a contradiction

Since all vertices eventually end up in S, get correctness of the algorithm □

Linear Programming

- Given an $m \times n$ matrix A and a size-m vector b and a size-n vector c, find a vector x of n elements that maximizes $\sum_{i=1}^{n} c_i x_i$ subject to $Ax \leq b$
- E.g., $c = \begin{bmatrix} 2 & -3 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ -1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 22 \\ 4 \\ -8 \end{bmatrix}$ implies:

  maximize $2x_1 - 3x_2$ **subject to**

$$\begin{aligned} x_1 + x_2 &\leq 22 \\ x_1 - 2x_2 &\leq 4 \\ x_1 &\geq 8 \end{aligned}$$

- **Solution:** $x_1 = 16$, $x_2 = 6$

Difference Constraints and Feasibility

- **Decision version of this problem:** No objective function to maximize; simply want to know if there exists a **feasible solution**, i.e., an x that satisfies $Ax \leq b$
- Special case is when each row of A has exactly one 1 and one -1, resulting in a set of **difference constraints** of the form

$$x_j - x_i \leq b_k$$

- **Applications:** Any application in which a certain amount of time must pass between events (x variables represent times of events)

Difference Constraints and Feasibility (2)

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 5 \\ 4 \\ -1 \\ -3 \\ -3 \end{bmatrix}$$

Difference Constraints and Feasibility (3)

Is there a setting for $x_1, \ldots, x_5$ satisfying:

$$\begin{aligned} x_1 - x_2 &\leq 0 \\ x_1 - x_5 &\leq -1 \\ x_2 - x_5 &\leq 1 \\ x_3 - x_1 &\leq 5 \\ x_4 - x_1 &\leq 4 \\ x_4 - x_3 &\leq -1 \\ x_5 - x_3 &\leq -3 \\ x_5 - x_4 &\leq -3 \end{aligned}$$

One solution: $x = (-5, -3, 0, -1, -4)$

Constraint Graphs

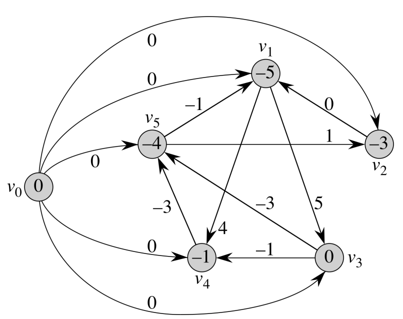
- Can represent instances of this problem in a **constraint graph** $G = (V, E)$
- Define a vertex for each variable, plus one more: If variables are $x_1, \ldots, x_n$, get $V = \{v_0, v_1, \ldots, v_n\}$
- Add a directed edge for each constraint, plus an edge from v_0 to each other vertex:

$$\begin{aligned} E &= \{(v_i, v_j) : x_j - x_i \leq b_k \text{ is a constraint}\} \\ &\quad \cup \{(v_0, v_1), (v_0, v_2), \ldots, (v_0, v_n)\} \end{aligned}$$

- Weight of edge (v_i, v_j) is b_k, weight of (v_0, v_ℓ) is 0 for all $\ell \neq 0$

Constraint Graph Example

$$\begin{aligned} x_1 - x_2 &\leq 0 \\ x_1 - x_5 &\leq -1 \\ x_2 - x_5 &\leq 1 \\ x_3 - x_1 &\leq 5 \\ x_4 - x_1 &\leq 4 \\ x_4 - x_3 &\leq -1 \\ x_5 - x_3 &\leq -3 \\ x_5 - x_4 &\leq -3 \end{aligned}$$



$(-5, -3, 0, -1, -4)$

Solving Feasibility with Bellman-Ford

Theorem: Let G be constraint graph for system of difference constraints. If G has a negative-weight cycle, then there is no feasible solution. If G has no negative-weight cycle, then a feasible solution is

$$x = [\delta(v_0, v_1), \delta(v_0, v_2), \ldots, \delta(v_0, v_n)]$$

- **Proof:** For any edge $(v_i, v_j) \in E$, triangle inequality says $\delta(v_0, v_j) \leq \delta(v_0, v_i) + w(v_i, v_j)$, so $\delta(v_0, v_j) - \delta(v_0, v_i) \leq w(v_i, v_j)$

$\Rightarrow$ $x_i = \delta(v_0, v_i)$ and $x_j = \delta(v_0, v_j)$ satisfies constraint $x_i - x_j \leq w(v_i, v_j)$

- If there is a negative-weight cycle $c = \langle v_i, v_{i+1}, \ldots, v_k = v_i \rangle$, then there is a system of inequalities $x_{i+1} - x_i \leq w(v_i, v_{i+1})$, $x_{i+2} - x_{i+1} \leq w(v_{i+1}, v_{i+2})$, $\ldots$, $x_k - x_{k-1} \leq w(v_{k-1}, v_k)$. Summing both sides gives $0 \leq w(c) < 0$, implying that a negative-weight cycle indicates no solution □

Can solve with Bellman-Ford in time $O(n^2 + nm)$