Computer Science & Engineering 423/823
Design and Analysis of Algorithms

Lecture 02 — Sorting Lower Bound (Section 8.1)

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Introduction

- Impossibility of algorithms: There are some problems that cannot be solved
  - We'll visit this throughout the semester, especially with NP-completeness
  - Today's example: there does not exist a general-purpose (**comparison-based**) algorithm to sort n elements in time $o(n \log n)$
  - Will show this by proving an $\Omega(n \log n)$ **lower bound** on comparison-based sorting

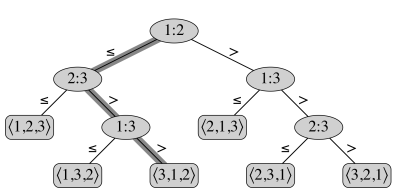
Comparison-Based Sorting Algorithms

- What is a comparison-based sorting algorithm?
  - The sorted order it determines is based **only** on comparisons between the input elements
  - E.g., Insertion Sort, Selection Sort, Mergesort, Quicksort, Heapsort
- What is **not** a comparison-based sorting algorithm?
  - The sorted order it determines is based on additional information, e.g., bounds on the range of input values
  - E.g., Counting Sort, Radix Sort

Decision Trees

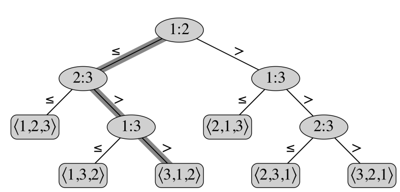
- A **decision tree** is a full binary tree that represents comparisions between elements performed by a particular sorting algorithm operating on a certain-sized input (n elements)
- **Key point:** a tree represents algorithm's behavior on *all possible inputs* of size n
- Each internal node represents one comparison made by algorithm
  - Each node labeled as $i : j$, which represents comparison $A[i] \leq A[j]$
  - If, in the particular input, it is the case that $A[i] \leq A[j]$, then control flow moves to left child, otherwise to the right child
  - Each leaf represents a possible output of the algorithm, which is a permutation of the input
  - All permutations must be in the tree in order for algorithm to work properly

Example for Insertion Sort



- If $n = 3$, Insertion Sort first compares $A[1]$ to $A[2]$
- If $A[1] \leq A[2]$, then compare $A[2]$ to $A[3]$
- If $A[2] > A[3]$, then compare $A[1]$ to $A[3]$
- If $A[1] \leq A[3]$, then sorted order is $A[1]$, $A[3]$, $A[2]$

Example for Insertion Sort (2)



- Example: $A = [7, 8, 4]$
- First compare 7 to 8, then 8 to 4, then 7 to 4
- Output permutation is $\langle 3, 1, 2 \rangle$, which implies sorted order is 4, 7, 8

Proof of Lower Bound

- Length of path from root to output leaf is number of comparisons made by algorithm on that input
- Worst-case number of comparisons is length of longest path (= **height** h)
- Number of leaves in tree is $n!$
- A binary tree of height h has at most 2^h leaves
- Thus we have $2^h \geq n! \geq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$
- Take base-2 logs of both sides to get

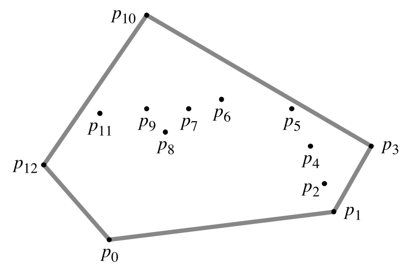
$$h \geq \lg \sqrt{2\pi} + (1/2)\lg n + n \lg n - n \lg e = \Omega(n \log n)$$

⇒ **Every** comparison-based sorting algorithm has an input that forces it to make $\Omega(n \log n)$ comparisons

⇒ Mergesort and Heapsort are *asymptotically optimal*

Another Lower Bound: Convex Hull

- Can use the lower bound on sorting to get a lower bound on the *convex hull* problem:
  - Given a set $Q \in \{p_1, p_2, \ldots, p_n\}$ of n points, each from $\mathbb{R}^2$, output $\mathrm{CH}(Q)$, which is the smallest convex polygon P such that each point from Q is on P's boundary or in its interior



Another Lower Bound: Convex Hull (cont'd)

- We will *reduce* the problem of sorting to that of finding a convex hull
- I.e., given any instance of the sorting problem $A = \{x_1, \ldots, x_n\}$, we will transform it to an instance of convex hull such that the time complexity of the new algorithm sorting will be no more than that of convex hull
  - ⇒ If convex hull could be solved in time $o(n \log n)$ then so can sorting
  - ⇒ Since that cannot happen, we know that convex hull is $\Omega(n \log n)$
- The reduction: transform A to $Q = \{(x_1, x_1^2), (x_2, x_2^2), \ldots, (x_n, x_n^2)\}$
  - ⇒ Takes $O(n)$ time
- Since the points on Q are on a parabola, all points of Q are on $\mathrm{CH}(Q)$
  - ⇒ Can read off the points of $\mathrm{CH}(Q)$ in $O(n)$ time
  - ⇒ Yields a sorted list of points from (*any*) A
- Time to sort A is $O(n)+$ convex hull $+O(n)$
- If time for convex hull is $o(n \log n)$, then sorting is $o(n \log n)$
  - ⇒ Convex hull time complexity is $\Omega(n \log n)$