Computer Science & Engineering 423/823 Design and Analysis of Algorithms

Lecture 01 — Medians and Order Statistics (Chapter 9)

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(Adapted from Vinodchandran N. Variyam)

Introduction

- Given an array A of n distinct numbers, the ith **order statistic** of A is its ith smallest element
  - $i = 1 \Rightarrow$ minimum
  - $i = n \Rightarrow$ maximum
  - $i = \lfloor (n+1)/2 \rfloor \Rightarrow$ (lower) median
- E.g. if $A = [8, 5, 3, 10, 4, 12, 6]$ then min = 3, max = 12, median = 6, 3rd order stat = 5
- **Problem:** Given array A of n elements and a number $i \in \{1, \ldots, n\}$, find the ith order statistic of A
- There is an obvious solution to this problem. What is it? What is its time complexity?
  - Can we do better? What if we only focus on $i = 1$ or $i = n$?

Finding Minimum

$small = A[1]$
for $i = 2$ *to* n **do**
    if $small > A[i]$ **then**
        $small = A[i]$
end
return $small$

Algorithm 1: Minimum(A, n)

Efficiency of Minimum(A)

- Loop is executed $n-1$ times, each with one comparison
  $\Rightarrow$ Total $n-1$ comparisons
- Can we do better?
- **Lower Bound:** Any algorithm finding minimum of n elements will need at least $n-1$ comparisons
  - Proof of this comes from fact that no element of A can be considered for elimination as the minimum until it's been compared at least once

Correctness of Minimum(A)

- Observe that the algorithm always maintains the **invariant** that at the end of each loop iteration, $small$ holds the minimum of $A[1 \cdots i]$
  - Easily shown by induction
- Correctness follows by observing that $i == n$ before **return** statement

Simultaneous Minimum and Maximum

- Given array A with n elements, find both its minimum and maximum
- What is the obvious algorithm? What is its (non-asymptotic) time complexity?
- Can we do better?

Simultaneous Minimum and Maximum

$large = \max(A[1], A[2])$
$small = \min(A[1], A[2])$
for $i = 2$ *to* $\lfloor n/2 \rfloor$ **do**
  $large = \max(large, \max(A[2i-1], A[2i]))$
  $small = \min(small, \min(A[2i-1], A[2i]))$
end
if n *is odd* **then**
  $large = \max(large, A[n])$
  $small = \min(small, A[n])$
return $(large, small)$

Algorithm 2: MinAndMax(A, n)

Explanation of MinAndMax

- Idea: For each pair of values examined in the loop, compare them directly
- For each such pair, compare the smaller one to $small$ and the larger one to $large$
- Example: $A = [8, 5, 3, 10, 4, 12, 6]$

Efficiency of MinAndMax

- How many comparisons does MinAndMax make?
- Initialization on Lines 1 and 2 requires only one comparison
- Each iteration through the loop requires one comparison between $A[2i-1]$ and $A[2i]$ and then one comparison to each of $large$ and $small$, for a total of three
- Lines 8 and 9 require one comparison each
- Total is at most $1 + 3(\lfloor n/2 \rfloor - 1) + 2 \leq 3\lfloor n/2 \rfloor$, which is better than $2n - 3$ for finding minimum and maximum separately

Selection of the ith Smallest Value

- Now to the general problem: Given A and i, return the ith smallest value in A
- Obvious solution is sort and return ith element
- Time complexity is $\Theta(n \log n)$
- Can we do better?

Selection of the ith Smallest Value (2)

- New algorithm: Divide and conquer strategy
- Idea: Somehow discard a constant fraction of the current array after spending only linear time
  - If we do that, we'll get a better time complexity
  - More on this later
- Which fraction do we discard?

Procedure Select

if $p == r$ **then**
    return $A[p]$
$q =$ Partition(A, p, r) // Like Partition in Quicksort
$k = q - p + 1$ // Size of $A[p \cdots q]$
if $i == k$ **then**
    return $A[q]$ // Pivot value is the answer
else if $i < k$ **then**
    return Select$(A, p, q - 1, i)$ // Answer is in left subarray
else
    return Select$(A, q + 1, r, i - k)$ // Answer is in right subarray

Algorithm 3: Select(A, p, r, i), which returns ith smallest element from $A[p \cdots r]$

What is Select Doing?

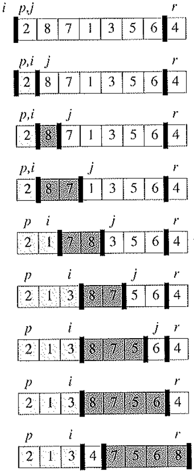
- Like in Quicksort, Select first calls Partition, which chooses a **pivot element** q, then reorders A to put all elements $< A[q]$ to the left of $A[q]$ and all elements $> A[q]$ to the right of $A[q]$
- E.g. if $A = [1, 7, 5, 4, 2, 8, 6, 3]$ and pivot element is 5, then result is $A' = [1, 4, 2, 3, 5, 7, 8, 6]$
- If $A[q]$ is the element we seek, then return it
- If sought element is in left subarray, then recursively search it, and ignore right subarray
- If sought element is in right subarray, then recursively search it, and ignore left subarray

Partitioning the Array

$x =$ ChoosePivotElement(A, p, r) // Returns index of pivot
exchange $A[x]$ with $A[r]$
$i = p - 1$
for $j = p$ *to* $r - 1$ **do**
    if $A[j] \leq A[r]$ **then**
        $i = i + 1$
        exchange $A[i]$ with $A[j]$
end
exchange $A[i + 1]$ with $A[r]$
return $i + 1$

Algorithm 4: Partition(A, p, r), which chooses a pivot element and partitions $A[p \cdots r]$ around it

Partitioning the Array: Example (Fig 7.1)



Compare each element $A[j]$ to x $(= 4)$ and swap with $A[i]$ if $A[j] \leq x$

Choosing a Pivot Element

- Choice of pivot element is critical to low time complexity
- Why?
- What is the best choice of pivot element to partition $A[p \cdots r]$?

Choosing a Pivot Element (2)

- Want to pivot on an element that it as close as possible to being the median
- Of course, we don't know what that is
- Will do **median of medians** approach to select pivot element

Median of Medians

- Given (sub)array A of n elements, partition A into $m = \lfloor n/5 \rfloor$ groups of 5 elements each, and at most one other group with the remaining $n \mod 5$ elements
- Make an array $A' = [x_1, x_2, \ldots, x_{m+1}]$, where x_i is median of group i, found by sorting (in constant time) group i
- Call Select$(A', 1, m+1, \lfloor (m+1)/2 \rfloor)$ and use the returned element as the pivot

Example

Split into teams, and work this example on the board: Find the 4th smallest element of $A = [4, 9, 12, 17, 6, 5, 21, 14, 8, 11, 13, 29, 3]$

Show results for each step of Select, Partition, and ChoosePivotElement

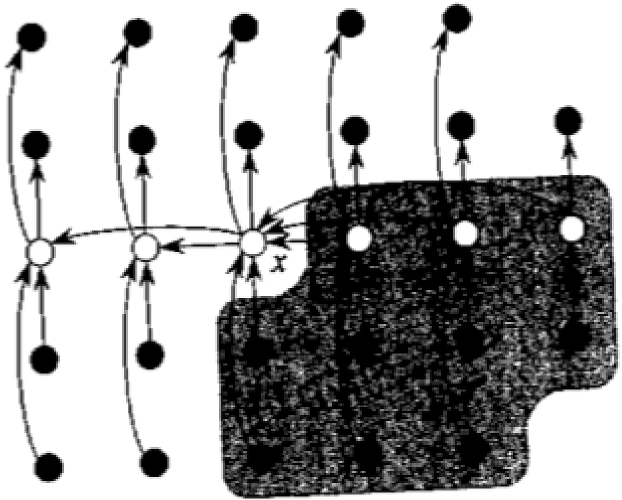
Time Complexity

- Key to time complexity analysis is lower bounding the fraction of elements discarded at each recursive call to Select
- On next slide, medians and median (x) of medians are marked, arrows indicate what is guaranteed to be greater than what
- Since x is less than at least half of the other medians (ignoring group with < 5 elements and x's group) and each of those medians is less than 2 elements, we get that the number of elements x is less than is at least

$$3\left(\left\lceil \frac{1}{2}\left\lceil \frac{n}{5} \right\rceil \right\rceil - 2\right) \geq \frac{3n}{10} - 6 \geq n/4 \qquad \text{(if } n \geq 120\text{)}$$

- Similar argument shows that at least $3n/10 - 6 \geq n/4$ elements are less than x
- Thus, if $n \geq 120$, each recursive call to Select is on at most $3n/4$ elements

Time Complexity (2)



Time Complexity (3)

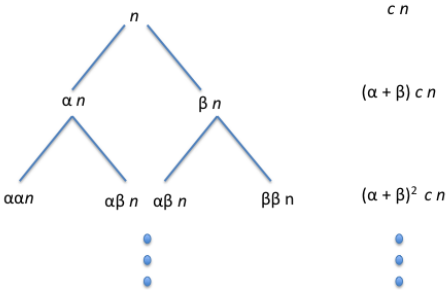
- Now can develop a **recurrence** describing Select's time complexity
- Let $T(n)$ represent total time for Select to run on input of size n
- Choosing a pivot element takes time $O(n)$ to split into size-5 groups and time $T(n/5)$ to recursively find the median of medians
- Once pivot element chosen, partitioning n elements takes $O(n)$ time
- Recursive call to Select takes time at most $T(3n/4)$
- Thus we get

$$T(n) \leq T(n/5) + T(3n/4) + O(n)$$

- Can express as $T(\alpha n) + T(\beta n) + O(n)$ for $\alpha = 1/5$ and $\beta = 3/4$
- **Theorem:** For recurrences of the form $T(\alpha n) + T(\beta n) + O(n)$ for $\alpha + \beta < 1$, $T(n) = O(n)$
- Thus Select has time complexity $O(n)$

Proof of Theorem

Top $T(n)$ takes $O(n)$ time ($= cn$ for some constant c). Then calls to $T(\alpha n)$ and $T(\beta n)$, which take a total of $(\alpha + \beta)cn$ time, and so on.



Summing these infinitely yields (since $\alpha + \beta < 1$)

$$cn(1 + (\alpha + \beta) + (\alpha + \beta)^2 + \cdots) = \frac{cn}{1 - (\alpha + \beta)} = c'n = O(n)$$

Master Method

- Another useful tool for analyzing recurrences
- **Theorem:** Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined as $T(n) = aT(n/b) + f(n)$. Then $T(n)$ is bounded as follows.
  1. If $f(n) = O(n^{\log_b a - \epsilon})$ for constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
  2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
  3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for constant $c < 1$ and sufficiently large n, then $T(n) = \Theta(f(n))$
- E.g. for Select, can apply theorem on $T(n) < 2T(3n/4) + O(n)$ (note the slack introduced) with $a = 2$, $b = 4/3$, $\epsilon = 1.4$ and get $T(n) = O\left(n^{\log_{4/3} 2}\right) = O\left(n^{2.41}\right)$

$\Rightarrow$ Not as tight for this recurrence