General Summation Formulas

Summation is linear:

\[
\sum_{i=a}^{b-1} cf(i) = c \sum_{i=a}^{b-1} f(i)
\]

\[
\sum_{i=a}^{b-1} (f(i) + g(i)) = \sum_{i=a}^{b-1} f(i) + \sum_{i=a}^{b-1} g(i)
\]

\[
\sum_{i=a}^{b-1} \Theta(f(i)) = \Theta\left( \sum_{i=a}^{b-1} f(i) \right)
\]

Summations can be broken into or assembled from disjoint intervals:

\[
\sum_{i=a}^{c-1} f(i) = \sum_{i=a}^{b-1} f(i) + \sum_{i=b}^{c-1} f(i)
\]

Summations of monotonic functions can be approximated by integration:

\[
\sum_{i=a}^{b-1} f(i) = O\left( \int_{a}^{b} f(x) \, dx \right) \quad \text{(if } f \text{ is monotonically increasing)}
\]

\[
\sum_{i=a}^{b-1} f(i) = \Theta\left( \int_{a}^{b} f(x) \, dx \right) \quad \text{(if } f \text{ is monotonically increasing and at most exponential)}
\]

Specific Summation Formulas

The sum of a constant is proportional to the length of the interval:

\[
\sum_{i=a}^{b-1} 1 = i|_{a}^{b}
\]

The sum of a linear term follows Euler’s formula:

\[
\sum_{i=a}^{b-1} i = \frac{i(i - 1)}{2}|_{a}^{b}
\]

The sum of an exponential term is an exponential term of the same order:

\[
\sum_{i=a}^{b-1} r^i = \frac{r^i}{r-1}|_{a}^{b}
\]
Advanced Topic: Umbral Calculus on Polynomials

The notation \( n(d) \), where an expression has a subscript written in parentheses, is a *Pochhammer symbol* for a *falling factorial*. Similar to how \( n^d \) means \( (n)(n-1)(n-2)\cdots \) \( d \) terms,

the expression \( n(d) \) means

\[
\underbrace{(n)(n-1)(n-2)\cdots}_{d\text{ terms}}
\]

So, for example, \( 4(3) = (4)(3)(2) = 24 \).

We can always write a polynomial in terms of falling factorials by repeatedly finding the leading term of the unrewritten portion and adding and subtracting a falling factorial with the same coefficient and degree. For instance, to rewrite \( 4i^3 \), we first add and subtract \( 4i(3) \):

\[
4i(3) + (4i^3 - 4i(3)) = 4i(3) + 12i^2 - 8i.
\]

then add and subtract \( 12i(2) \):

\[
4i(3) + 12i(2) + (12i^2 - 8i - 12i(2)) = 4i(3) + 12i(2) + 4i.
\]

and finally add and subtract \( 4i(1) \):

\[
4i(3) + 12i(2) + 4i(1) + (4i - 4i(1)) = 4i(3) + 12i(2) + 4i(1).
\]

The rule for summing falling factorials mimics the rule for integrating monomials in that

\[
\sum_{i=a}^{b-1} i(d) = \frac{i(d+1)}{d+1} \bigg|_a^b.
\]

(An easy way to prove this is to notice that \( \frac{i(d)}{d} \) is the formula for the \( d^{th} \) diagonal of Pascal’s triangle.)

Thus, to find the summation of an arbitrary polynomial, we can write it in terms of falling factorials and apply the rule above. For instance,

\[
\sum_{i=0}^{n-1} 4i^3 = \sum_{i=0}^{n-1} \left( 4i(3) + 12i(2) + 4i(1) \right)
\]

\[
= \left( 4 \frac{i(4)}{4} + 12 \frac{i(3)}{3} + 4 \frac{i(2)}{2} \right) \bigg|_0^n
\]

\[
= (i^4 - 2i^3 + i^2) \bigg|_0^n
\]

\[
= n^4 - 2n^3 + n^2.
\]