Interdependent Defense Games: Modeling Interdependent Security under Deliberate Attacks (Supplementary Material)

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A Proof of Proposition 1

First note that Assumption 1 considerably simplifies some of the expressions involving external risk/safety. This is because any pure strategy in $\mathcal{B}$ is either a vector of all 0's, or exactly one 1. For instance, in this case we have

$$s_i(\mathbf{a}_{\mathrm{Pa}(i)}, \mathbf{b}_{\mathrm{Pa}(i)}) = \begin{cases} \sum_{j \in \mathrm{Pa}(i)} b_j e_{ij}(a_j, 1), & \text{if } b_k = 1 \text{ for some } k \in \mathrm{Pa}(i), \\ 1, & \text{if } b_k = 0 \text{ for all } k \in \mathrm{Pa}(i), \end{cases}$$
$$= 1 - \sum_{j \in \mathrm{Pa}(i)} b_j (1 - a_j) \widehat{q}_{ji},$$

so that

$$r_i(\mathbf{a}_{\mathrm{Pa}(i)}, \mathbf{b}_{\mathrm{Pa}(i)}) = \sum_{j \in \mathrm{Pa}(i)} b_j (1 - a_j) \widehat{q}_{ji},$$

and

$$b_i\, s_i(\mathbf{a}_{\mathrm{Pa}(i)}, \mathbf{b}_{\mathrm{Pa}(i)}) = b_i.$$

Also, if the IDD game has a PSNE $(\mathbf{a}^*, \mathbf{b}^*)$, then the attacker's payoff in it is

$$U(\mathbf{a}^*, \mathbf{b}^*) = \left[\max_{i\in[n]}(1-a_i^*)\left(\widehat{p}_i L_i + \sum_{j\in\mathrm{Ch}(i)} \widehat{q}_{ij}(a_j^*\alpha_j + (1-a_j^*))L_j\right) - C_i^0\right]^+$$

where for any real number $z \in \mathbb{R}$, the operator $[z]^+ \equiv \max(z, 0)$; in addition, if $b_k^* = 1$ for some $k \in [n]$, then

$$(1-a_k^*)\left(\widehat{p}_k L_k + \sum_{j\in\mathrm{Ch}(k)} \widehat{q}_{kj}(a_j^*\alpha_j + (1-a_j^*))L_j\right) - C_k^0 \geq$$
$$\left[\max_{i\in[n]}(1-a_i^*)\left(\widehat{p}_i L_i + \sum_{j\in\mathrm{Ch}(i)} \widehat{q}_{ij}(a_j^*\alpha_j + (1-a_j^*))L_j\right) - C_i^0\right]^+ . \quad (1)$$

The proof of the proposition is by contradiction. Consider an IDD game that satisfies the conditions of the proposition. Let $(\mathbf{a}^*, \mathbf{b}^*)$ be a PSNE of the game. We need to consider two cases at the PSNE: (1) there is some attack and (2) there is no attack.

1. If there is some attack, then $b_k^* = 1$ for some site $k \in [n]$, and for all $i \neq k$, $b_i^* = 0$. In addition, because $\mathbf{b}^*$ is consistent with the aggressor's best response to $\mathbf{a}^*$, we have, using condition 1 above,

   $$(1-a_k^*)\left(\widehat{p}_k L_k + \sum_{j\in\mathrm{Ch}(k)} \widehat{q}_{kj}(a_j^*\alpha_j + (1-a_j^*))L_j\right) \geq C_k^0 > 0 ,$$

   The last condition and Assumption 3 implies $a_k^* = 0$. Hence, by the best-response condition of site k, we have

   $$C_k + \alpha_k r_k(\mathbf{a}^*_{\mathrm{Pa}(k)}, \mathbf{b}^*_{\mathrm{Pa}(k)})L_k \geq \widehat{p}_k L_k + (1-\widehat{p}_k) r_k(\mathbf{a}^*_{\mathrm{Pa}(k)}, \mathbf{b}^*_{\mathrm{Pa}(k)})L_k .$$

   Because the attack occurs at k, the transfer risk $r_k(\mathbf{a}^*_{\mathrm{Pa}(k)}, \mathbf{b}^*_{\mathrm{Pa}(k)}) = r_k(\mathbf{a}^*_{\mathrm{Pa}(k)}, \mathbf{0}) = 0$ at the PSNE. Therefore, the last condition simplifies to

   $$C_k \geq \widehat{p}_k L_k ,$$

   which contradicts Assumption 2.

2. If there is no attack, then $\mathbf{b}^* = \mathbf{0}$. In this case, the site's best-response conditions imply $\mathbf{a}^* = \mathbf{0}$. From the attacker's best-response condition we obtain

$$\widehat{p}_k L_k + \sum_{j \in \mathrm{Ch}(k)} \widehat{q}_{kj} L_j \leq C_k^0 \ ,$$

which contradicts Assumption 3.

B Proof of Proposition 2

Throughout this proof, by the hypothesis of the proposition, we assume we are dealing with single-simultaneous-attack transfer-vulnerable IDD games. We also use the same notation as that introduced before the statement of the proposition in the main text.

First recall that Assumption 1, in the context of mixed strategies, implies the probability of no attack $y_0 \equiv 1 - \sum_i^n y_i$. This is because under this assumption

$$y(\mathbf{b}) = \begin{cases} y_{B_i}(b_i) = y_i, & \text{if } b_i = 1 \text{ for } \textit{exactly one } i \in [n], \\ y_0, & \text{if } b_i = 0 \text{ for } \textit{all } i \in [n], \\ 0, & \text{otherwise.} \end{cases}$$

Recall also that, when used in combination, Assumptions 1 and 4 greatly simplify the best-response condition of the internal players because now

$\widehat{s}_i(\mathbf{x}_{\mathrm{Pa}(i)}, y_{\mathrm{PF}(i)}) = y_i$. In particular, we have [1]

$$
\begin{aligned}
s_i(\mathbf{x}_{\mathrm{Pa}(i)}, y_{\mathrm{Pa}(i)}) &\equiv \sum_{\mathbf{b}_{\mathrm{Pa}(i)}} y_{\mathrm{Pa}(i)}(\mathbf{b}_{\mathrm{Pa}(i)}) s_i(\mathbf{x}_{\mathrm{Pa}(i)}, \mathbf{b}_{\mathrm{Pa}(i)}) \\
&= \sum_{\mathbf{b}_{\mathrm{Pa}(i)}} y_{\mathrm{Pa}(i)}(\mathbf{b}_{\mathrm{Pa}(i)}) \prod_{j \in \mathrm{Pa}(i)} e_{ij}(x_j, b_j) \\
&= \left(y_0 + \sum_{j \in [n] - \mathrm{Pa}(i)} y_j \right) + \sum_{j \in \mathrm{Pa}(i)} y_j e_{ij}(x_j, 1) \\
&= \left(y_0 + \sum_{j \in [n] - \mathrm{Pa}(i)} y_j \right) + \sum_{j \in \mathrm{Pa}(i)} y_j (x_j + (1 - x_j)(1 - \widehat{q}_{ji})) \\
&= \left(y_0 + \sum_{j \in [n] - \mathrm{Pa}(i)} y_j \right) + \sum_{j \in \mathrm{Pa}(i)} y_j (x_j + (1 - x_j) - (1 - x_j)\widehat{q}_{ji}) \\
&= \left(y_0 + \sum_{j \in [n] - \mathrm{Pa}(i)} y_j \right) + \sum_{j \in \mathrm{Pa}(i)} y_j (1 - (1 - x_j)\widehat{q}_{ji}) \\
&= \left(y_0 + \sum_{j \in [n] - \mathrm{Pa}(i)} y_j \right) + \sum_{j \in \mathrm{Pa}(i)} y_j - \sum_{j \in \mathrm{Pa}(i)} y_j (1 - x_j)\widehat{q}_{ji} \\
&= 1 - \sum_{j \in \mathrm{Pa}(i)} y_j (1 - x_j)\widehat{q}_{ji} \ ,
\end{aligned}
$$

so that $r_i(\mathbf{x}_{\mathrm{Pa}(i)}, y_{\mathrm{Pa}(i)}) = \sum_{j \in \mathrm{Pa}(i)} y_j (1 - x_j)\widehat{q}_{ji}$, and

$$
\begin{aligned}
f_i(\mathbf{x}_{\mathrm{Pa}(i)}, y_{\mathrm{Pa}(i)}) &\equiv \sum_{\mathbf{b}_{\mathrm{PF}(i)}} y_{\mathrm{PF}(i)}(\mathbf{b}_{\mathrm{PF}(i)})\; b_i\; s_i(\mathbf{x}_{\mathrm{Pa}(i)}, \mathbf{b}_{\mathrm{Pa}(i)}) \\
&= \sum_{\mathbf{b}_{\mathrm{PF}(i)}} y_{\mathrm{PF}(i)}(\mathbf{b}_{\mathrm{PF}(i)})\; b_i \prod_{j \in \mathrm{Pa}(i)} e_{ij}(x_j, b_j) \\
&= \left(y_0 + \sum_{j \in [n] - \mathrm{PF}(i)} y_j \right) \times 0 \times 1 + y_i + \sum_{j \in \mathrm{Pa}(i)} y_j \times 0 \times e_{ij}(x_j, 1) \\
&= y_i \ .
\end{aligned}
$$

Combining the last derivation above with Assumption 4 (i.e., $\alpha_i = 1$) leads

[1] Note that $e_{ij}(x_j, 0) = 1$.

to

$$\widehat{s}_i(\mathbf{x}_{\mathrm{Pa}(i)}, y_{\mathrm{PF}(i)}) \equiv f_i(\mathbf{x}_{\mathrm{Pa}(i)}, y_{\mathrm{PF}(i)}) + \frac{1-\alpha_i}{\widehat{p}_i} r_i(x_{\mathrm{Pa}(i)}, y_{\mathrm{Pa}(i)}) = y_i \ ,$$

as claimed above. Hence, the best-response $\mathcal{BR}_i$ of defender i *directly* depends on y_i *only* (i.e., $\mathcal{BR}_i$ is *conditionally* independent of the mixed strategies $\mathbf{x}_{\mathrm{Pa}(i)}$ of its parent nodes $\mathrm{Pa}(i)$ of defender node i in the network *given* the probability y_i that the attacker's mixed-strategy y assigns to a direct attack to i); thus, in what follows, we abuse notation and define

$$\mathcal{BR}_i(y_i) \equiv \mathcal{BR}_i(\mathbf{x}_{\mathrm{Pa}(i)}, y_{\mathrm{PF}(i)}) = \begin{cases} \{1\}, & \text{if } y_i > \widehat{\Delta}_i, \\ \{0\}, & \text{if } y_i < \widehat{\Delta}_i, \\ [0,1], & \text{if } y_i = \widehat{\Delta}_i. \end{cases}$$

Next, we prove some useful properties of the MSNE. [2]

Claim 1. *In every MSNE* $(\mathbf{x}, y)$*, for all* $i \in [n]$*, if the probability of a direct attack to a defender* i *is* $y_i = 0$ *then the probability of investment of defender* i *is* $x_i = 0$*. In addition, if* $y_i = 0$ *for some* $i \in [n]$ *then the probability of no attack* $y_0 = 0$*.*

Proof. By $\mathcal{BR}_i$, $y_i = 0 < \widehat{\Delta}_i$ implies $x_i = 0$. For the second part, if $y_i = 0$ for some defender $i \in [n]$, then, by $\mathcal{BR}_0$, we have

$$\max_t M_t^0(x_t) \geq M_i^0(x_i) = \overline{M}_k^0 > 0,$$

and thus $y_0 = 0$. □

Proposition B.1. *In every MSNE* $(\mathbf{x}, y)$*, an attack is always possible:* $y_0 < 1$*.*

Proof. The proof is by contradiction. Let $(\mathbf{x}, y)$ be an MSNE. Suppose there is no attack: $y_0 = 1$. Then, $\sum_{i=1}^n y_i = 1 - y_0 = 0$, so that $y_i = 0$ for all $i \in [n]$. Because $y_i = 0$ for some $i \in [n]$, Claim 1 yields $y_0 = 0$, a contradiction. □

Lemma 1. *In every MSNE* $(\mathbf{x}, y)$*, the probability* y_i *of direct attack to defender* i *is no larger than* $\widehat{\Delta}_i < 1$*.*

[2] Throughout the proof, to simplify notation, we drop the '*' superscript used in the main text to denote MSNE.

Proof. The proof is by contradiction. Suppose there is some MSNE in which $y_i > \widehat{\Delta}_i$ for some $i \in [n]$. Then, $x_i = 1$ and in turn $M_i^0(1) = -C_i^0 < 0$. Because the attacker can always achieve expected payoff 0 by not attacking anyone, the last condition implies $y_i = 0$, a contradiction. □

Claim 2. *Let y be the mixed-strategy of the attacker in some MSNE. If the probability of no attack $y_0 > 0$, then the probability of direct attack to defender i is equal to the cost-to-conditional expected-loss of defender i: $y_i = \widehat{\Delta}_i$ for all $i \in [n]$.*

Proof. The proof is by contradiction. By Lemma 1 $y_i \leq \widehat{\Delta}_i$ for all $i \in [n]$. Suppose $y_i < \widehat{\Delta}_i$ for some i. Then, by $\mathcal{BR}_i$, we have $x_i = 0$, and by $\mathcal{BR}_0$, we have $0 \geq \overline{M}_i^0 > 0$, a contradiction. □

Lemma 2. *In every MSNE $(\mathbf{x}, y)$ of an IDD game in which the total of cost-to-conditional expected-loss of all defenders is $\sum_{i=1}^n \widehat{\Delta}_i < 1$, there may not be an attack: $y_0 > 0$.*

Proof. By Lemma 1, $y_i \leq \widehat{\Delta}_i$ for all $i \in [n]$. Using the last statement, note that

$$1 - y_0 = \sum_{i=1}^{n} y_i \leq \sum_{i=1}^{n} \widehat{\Delta}_i < 1,$$

from which the lemma immediately follows. □

As stated in the main text, we partition the class of IDD games into three subclasses, based on whether $\sum_{i=1}^n \widehat{\Delta}_i$ is (1) less than, (2) equal to, or (3) greater than 1. We consider each subclass in turn.

Proposition B.2. *The joint mixed-strategy $(\mathbf{x}, y)$ is an MSNE of an IDD game in which the total cost-to-conditional expected-loss of all defenders is $\sum_{i=1}^n \widehat{\Delta}_i < 1$ if and only if it satisfies the following properties.*

1. *There may not be an attack with probability of no attack equal to one minus the cost-to-conditional expected-loss of all defenders: for all defenders i $1 > y_0 = 1 - \sum_{i=1}^n \widehat{\Delta}_i > 0$.*

2. *Every defender has non-zero chance of being attacked directly, and this probability equals the respective defender's cost-to-conditional expected-loss of defender: for all defenders $i \in [n]$, $y_i = \widehat{\Delta}_i > 0$.*

3. *Every defender invests some but none does fully, and in particular, the probability a defender does* not *invest equals the respective cost-to-loss ratio to the attacker: for all defenders $i \in [n]$, $0 < x_i = 1 - \eta_i^0 < 1$.*

Proof. Suppose the joint mixed-strategy $(\mathbf{x}, y)$ satisfies the properties above. Then, every defender is indifferent (i.e., for all $i \in [n]$, $\mathcal{BR}_i(y_i) = [0, 1]$, because $y_i = \widehat{\Delta}_i$), as is also the attacker (i.e., $\mathcal{BR}_0(\mathbf{x})$ equals the set of *all* probability distributions over $n + 1$ events because $M_i^0(x_i) = 0$ for all $i \in [n]$). Hence, $(\mathbf{x}, y)$ is an MSNE.

Now suppose $(\mathbf{x}, y)$ is an MSNE of the game. By Lemma 2, $y_0 > 0$. Hence, for all $i \in [n]$, we have $y_i = \widehat{\Delta}_i > 0$ by Claim 2. Both of the previous sentences together imply $M_i^0(x_i) = 0$ for all $i \in [n]$, because of $\mathcal{BR}_0$. Simple algebra yields that $x_i = 1 - \eta_i^0$. Finally, because $y_0 + \sum_{i=1}^n y_i = 1$, we have $y_0 = 1 - \sum_{i=1}^n \widehat{\Delta}_i$. □

Proposition B.3. *The joint mixed-strategy $(\mathbf{x}, y)$ is an MSNE of an IDD game in which $\sum_{i=1}^n \widehat{\Delta}_i = 1$ if and only if it satisfies the following properties.*

1. *There is always an attack: $y_0 = 0$.*

2. *Every defender has non-zero chance of being attacked directly, and this probability equals the respective defender's cost-to-conditional expected-loss of defender i: for all defenders $i \in [n]$, $y_i = \widehat{\Delta}_i > 0$.*

3. *No defender invests fully, and the possible investment probabilities are connected by a 1-d line segment in $\mathbb{R}^n$:*

$$x_i = 1 - \frac{v + C_i^0}{\overline{L}_i^0} \text{ for all } i \in [n]$$

with $0 \le v \le \min_{i \in [n]} \overline{M}_i^0$.

Proof. Suppose the joint mixed-strategy $(\mathbf{x}, y)$ satisfies the properties above. Then, every defender is indifferent: for all $i \in [n]$, $\mathcal{BR}_i(y_i) = [0, 1]$, because $y_i = \widehat{\Delta}_i$. To test $y \in \mathcal{BR}_0(\mathbf{x})$, note $0 \le (1 - x_i)\overline{L}_i^0 - C_i^0 = M_i^0(x_i) = \max_{t \in [n]} M_t^0(x_t)$ for all $i \in [n]$, and

$$\sum_{i=1}^n y_i M_i^0(x_i) = \sum_{i=1}^n y_i \max_{t \in [n]} M_t^0(x_t) =$$

$$\left(\sum_{i=1}^n y_i\right) \max_{t \in [n]} M_t^0(x_t) = \max_{t \in [n]} M_t^0(x_t).$$

Let the joint mixed-strategy $(\mathbf{x}, y)$ be an MSNE of the game. Let $I \equiv I(y) \equiv \{i \in [n] \mid y_i > 0\}$. Note that $y_k = 0$ for all $k \notin I$. We first prove the following lemma.

Lemma 3. *$I = [n]$.*

Proof. The proof is by contradiction. Suppose $I \neq [n]$. By Proposition B.1, $y_0 < 1 = y_0 + \sum_{i=1}^n y_i$ so that $y_i > 0$ for some $i \in [n]$, and therefore $I \neq \emptyset$. Also, there exists some $k \in [n] - I$, for which $y_k = 0$. By Claim 1, we then have for all $k \notin I$, $x_k = 0$. By $\mathcal{BR}_0$ and Assumption 3, for all $i, t \in I \neq \emptyset$ and $k \notin I$,

$$M_i^0(x_i) = M_t^0(x_t) \geq \overline{M}_k^0.$$

The condition above yields the following upper bound on the mixed strategies of the defenders in $i \in I$, after applying simple algebraic manipulations: for all $i \in I, k \notin I$,

$$x_i \leq 1 - \frac{\overline{M}_k^0 + C_i^0}{\overline{L}_i^0} < 1.$$

By $\mathcal{BR}_i$, this implies that $y_i \leq \widehat{\Delta}_i$ for all $i \in I$. Putting all of the above together, we have

$$1 = \sum_{i=0}^n y_i = \sum_{i=1}^n y_i = \sum_{i \in I} y_i \leq \sum_{i \in I} \widehat{\Delta}_i \leq \sum_{i=1}^n \widehat{\Delta}_i = 1.$$

Now, because $I \neq [n]$ (by the hypothesis assumed to obtain a contradiction), we have $\sum_{k \notin I} \widehat{\Delta}_k > 0$, and

$$\sum_{i \in I} y_i = \sum_{i=1}^n \widehat{\Delta}_i = \sum_{i \in I} \widehat{\Delta}_i + \sum_{k \notin I} \widehat{\Delta}_k > \sum_{i \in I} \widehat{\Delta}_i \geq \sum_{i \in I} y_i,$$

a contradiction. □

By the last lemma and $\mathcal{BR}_0$, we have

$$(1 - x_1)\overline{L}_1^0 - C_1 = \cdots = (1 - x_n)\overline{L}_n^0 - C_n \geq 0$$

Let $v \equiv (1 - x_1)\overline{L}_1^0 - C_1$. Then, $1 - x_i = \frac{v + C_i^0}{\overline{L}_i^0} > 0$. If $v > 0$ then $y_0 = 0$. Because $x_i < 1$, we have $y_i \leq \widehat{\Delta}_i$ for all $i \in [n]$. Thus, we have $y_i = \widehat{\Delta}_i$ for all $i \in [n]$ because otherwise if $y_t < \widehat{\Delta}_t$ for some $t \in [n]$, then $1 = y_0 + y_t + \sum_{i=1, i \neq t}^n y_i < \sum_{i=1}^n \widehat{\Delta}_i = 1$, a contradiction. If, instead, $v = 0$, for all i, we have $x_i = 1 - \eta_i^0 > 0$, which implies $y_i = \widehat{\Delta}_i$. Therefore, $y_0 = 1 - \sum_{i=1}^n y_i = 1 - \sum_{i=1}^n \widehat{\Delta}_i = 0$.

□

Lemma 4. *In every MSNE* $(\mathbf{x}, y)$ *of an IDD game in which* $\sum_{i=1}^{n} \widehat{\Delta}_i > 1$, *the probability of no attack* $y_0 = 0$.

Proof. The proof is by contradiction. Suppose $y_0 > 0$. Then, by Claim 2, we have $y_i = \widehat{\Delta}_i$ for all $i \in [n]$, and $1 = \sum_{i=0}^{n} y_i = \sum_{i=1}^{n} \widehat{\Delta}_i > 1$, a contradiction. □

Proposition B.4. *In every MSNE* $(\mathbf{x}, y)$ *of an IDD game, the probability of no attack* $y_0 > 0$ *if and only if the game has the property* $\sum_{i=1}^{n} \widehat{\Delta}_i < 1$.

Proof. The "if" part is Lemma 2. For the "only if" part, the case in which $\sum_{i=1}^{n} \widehat{\Delta}_i = 1$ follows from Proposition B.3; the case in which $\sum_{i=1}^{n} \widehat{\Delta}_i > 1$ follows from Lemma 4. □

Proposition B.5. *In every MSNE* $(\mathbf{x}, y)$ *of an IDD game in which* $\sum_{i=1}^{n} \widehat{\Delta}_i > 1$, *no defender is fully investing and some defender is not investing at all (i.e.,* $x_i = 0$ *for some* $i \in [n]$*).*

Proof. The proof is by contradiction. Proposition B.4 yields $y_0 = 0$. Suppose $x_i = 1$ for some $i \in [n]$. Then, by $\mathcal{BR}_i$, $y_i \geq \widehat{\Delta}_i$, and by $\mathcal{BR}_0$ and the fact that $y_0 = 0$, we have $0 > -C_i^0 = M_i(x_i) \geq 0$, which implies $y_i = 0$, a contradiction.

Now suppose $0 < x_i < 1$ for all $i \in [n]$. Then, by $\mathcal{BR}_i$, we have $y_i = \widehat{\Delta}_i$ for all $i \in [n]$. Thus we have $1 = \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \widehat{\Delta}_i > 1$, a contradiction. □

Proposition B.6. *The joint mixed-strategy* $(\mathbf{x}, y)$ *is an MSNE of an IDD game in which* $\sum_{i=1}^{n} \widehat{\Delta}_i > 1$ *if and only if it satisfies the following properties.*

1. *There is always an attack:* $y_0 = 0$.
2. *There exists a non-singleton, non-empty subset* $I \subset [n]$, *such that* $\min_{i \in I} \overline{M}_i^0 \geq \max_{k \notin I} \overline{M}_k^0$, *if* $I \neq [n]$, *and the following holds.*
   (a) *No defender outside* I *invests or is attacked directly:* $x_k = 0$ *and* $y_k = 0$ *for all* $k \notin I$.
   (b) *Let* $J \equiv \arg\min_{i \in I} \overline{M}_i^0$. *No defender in* J *invests and the probability of that defender being attacked directly is at most the defender's cost-to-expected-loss ratio: for all* $i \in J$, $x_i = 0$ *and* $0 \leq y_i \leq \widehat{\Delta}_i$; *in addition,* $\sum_{i \in J} y_i = 1 - \sum_{t \in I - J} \widehat{\Delta}_i$.

(c) *Every defender in $I - J$ partially invests and has positive probability of being attacked directly equal to the defender's cost-to-expected-loss ratio: for all $i \in I - J$, $y_i = \widehat{\Delta}_i$ and*

$$0 < x_i = 1 - \frac{\min_{t \in I} \overline{M}_t^0 + C_i^0}{\overline{L}_i^0} < 1.$$

Proof. For the "if" part, we need to show $(\mathbf{x}, y)$ form mutual best-responses. For all $k \notin I$, $x_k = 0 \in \mathcal{BR}_k(y)$ because $y_k = 0 < \widehat{\Delta}_k$. For all $j \in J$, $x_j = 0 \in \mathcal{BR}_j(y)$ because $y_j \leq \widehat{\Delta}_j$. Finally, for all $i \in I - J$, $x_i \in \mathcal{BR}_i(y_i) = [0, 1]$ because $y_i = \widehat{\Delta}_i$. Hence, we have $x_i \in \mathcal{BR}_i(y_i)$ for all $i \in [n]$. For the attacker, let $v \equiv v(I) \equiv \min_{i \in I} \overline{M}_i^0$. We have for all $k \notin I$, $M_k(x_k) = \overline{M}_k^0 \leq \max_{l \notin I} \overline{M}_l^0 \leq \min_{i \in I} \overline{M}_i^0 = v$, where the first equality holds because $x_k = 0$ and the second inequality by the properties of I. We also have for all $j \in J$, $M_j(x_j) = \overline{M}_j^0 = \min_{i \in I} \overline{M}_i^0 = v$, where the first equality holds because $x_j = 0$ and the second follows from the definition of J. Finally, using simple algebra, we also have for all $i \in I - J$,

$$\begin{aligned} M_i(x_i) &= (1 - x_i)\overline{L}_i^0 - C_i^0 \\ &= \left(\frac{\min_{t \in I} \overline{M}_t^0 + C_i^0}{\overline{L}_i^0} \right) \overline{L}_i^0 - C_i^0 \\ &= \min_{t \in I} \overline{M}_t^0 + C_i^0 - C_i^0 = \min_{t \in I} \overline{M}_t^0 = v. \end{aligned}$$

Hence, we have for all $i \in [n]$, $M_i(x_i) \leq v$. The expected payoff of the attacker under the given joint mixed-strategy is

$$\begin{aligned} \sum_{i=1}^{n} y_i M_i(x_i) &= \sum_{j \in J} y_j M_j(x_j) + \sum_{i \in I - J} y_i M_i(x_i) \\ &= \sum_{j \in J} y_j v + \sum_{i \in I - J} y_i v \\ &= v \left(\sum_{j \in J} y_j + \sum_{i \in I - J} y_i \right) \\ &= v \left(\sum_{i=1}^{n} y_i \right) = v \geq M_i(x_i), \end{aligned}$$

for all $i \in [n]$. Hence, we also have $y \in \mathcal{BR}_0(\mathbf{x})$, and the joint mixed-strategy $(\mathbf{x}, y)$ is an MSNE.

We now consider the "only if" part of the proposition. Let $(\mathbf{x}, y)$ be an MSNE and let $I \equiv I(y) \equiv \{i \in [n] \mid y_i > 0\}$ be the support of the aggressor's mixed strategy. We now show that I is a non-singleton and non-empty subset of $[n]$.

Claim 3. $1 < |I| \leq n$.

Proof. From Proposition B.1, we have $I \neq \emptyset$. That I is not a singleton set follows from Lemma 1. □

By Proposition B.4, we have $y_0 = 0$. Applying Proposition B.5, let $t \in [n]$ be such that $x_t = 0$. Also by Proposition B.5, the aggressor achieves a positive expected payoff: $\sum_{i=1}^n y_i M_i^0(x_i) = \max_{l=1}^n M_l^0(x_l) \geq M_t^0(x_t) = \overline{M}_t^0 > 0$. For all $k \notin I$, because $y_k = 0$, Claim 1 implies $x_k = 0$.

By $\mathcal{BR}_0$, if I is a strict, non-empty and non-singleton subset of $[n]$, we have, for all $i \in I$ and $k \notin I$,

$$\overline{M}_i^0 \geq M_i^0(x_i) = \max_{l \in I} M_l^0(x_l) \geq \overline{M}_k^0 > 0;$$

otherwise, if $I = [n]$, we have, for all $i \in [n]$,

$$M_i^0(x_i) = \max_{l \in [n]} M_l^0(x_l) = M_t^0(x_t) = \overline{M}_t^0 > 0.$$

Let $v \equiv v(I) \equiv \max_{l \in I} M_l^0(x_l)$. Then, the above expressions imply that for all $i \in I$, we have

$$0 < x_i = 1 - \frac{v + C_i^0}{\overline{L}_i^0} < 1.$$

In addition, we have that if I is a strict, non-empty and non-singleton subset of $[n]$, we have,

$$v = \overline{M}_t^0 \geq \min_{i \in I} \overline{M}_i^0 \geq v \geq \max_{k \notin I} \overline{M}_k^0;$$

and if, instead, $I = [n]$, then

$$v = \overline{M}_t^0 = \min_{i \in [n]} \overline{M}_i^0.$$

Hence, we have $v = \min_{i \in I} \overline{M}_i^0$.

Let $J \equiv J(I) \equiv \arg\min_{i \in I} \overline{M}_i^0$. For all $i \in J$, we have $\overline{M}_i^0 = v$, and thus

$$x_i = 1 - \frac{v + C_i^0}{\overline{L}_i^0} = 1 - \frac{\overline{M}_i^0 + C_i^0}{\overline{L}_i^0} = 1 - \frac{\overline{L}_i^0 - C_i^0 + C_i^0}{\overline{L}_i^0} = 0,$$

and by $\mathcal{BR}_i$, we have $0 \leq y_i \leq \widehat{\Delta}_i$.

For all $i \in I - J$, we have $\overline{M}_i^0 > v$, and thus

$$0 = 1 - \frac{\overline{M}_i^0 + C_i^0}{\overline{L}_i^0} < x_i = 1 - \frac{v + C_i^0}{\overline{L}_i^0} < 1,$$

and by $\mathcal{BR}_i$, we have $y_i = \widehat{\Delta}_i$.

Finally, we have $\sum_{i \in J} y_i = 1 - \sum_{i \in I-J} \widehat{\Delta}_i$, because y is a mixed strategy (i.e, a probability distribution). □

Hence, from the proof of the last proposition we can infer that if the $\overline{M}_l^0$'s form a complete order, then the last condition allows us to search for an MSNE by exploring only $n-2$ sets, as opposed to 2^{n-2} if done naively.

It turns out a complete order is not necessary. The following claim allows us to safely move all the defenders with the same value of $\overline{M}_i^0$ in a group as a whole inside or outside I.

Claim 4. *Let $I \subset [n]$, such that $I' \subset I$, $|I'| < |I| < n-1$. Suppose we find an MSNE $(\mathbf{x}, y)$ such that $I' = \{i \mid y_i > 0\}$, with the property that $\min_{l \in I'} \overline{M}_l^0 = \max_{k \notin I'} \overline{M}_k^0$. In addition, suppose I satisfies $\min_{l \in I'} \overline{M}_l^0 = \min_{l \in I} \overline{M}_l^0 \geq \max_{k \notin I} \overline{M}_k^0$. Then, we can also find $(\mathbf{x}, y)$ using partition I.*

Proof. To simplify the notation, let $v \equiv \min_{l \in I} \overline{M}_l^0 = \min_{l \in I'} \overline{M}_l^0$, $J' \equiv \arg\min_{l \in I'} \overline{M}_l^0$ and $J \equiv \arg\min_{i \in I} \overline{M}_i^0$. The hypothesis implies that $(\mathbf{x}, y)$ satisfies the following properties.

$$\begin{aligned}
\text{for all } i \notin I'\text{: } & x_i = y_i = 0 \\
\text{for all } i \in J'\text{: } & x_i = 0 \text{ and } 0 \leq y_i \leq \widehat{\Delta}_i; \\
& \text{also} \sum_{i \in J'} y_i = 1 - \sum_{i \in I' - J'} \widehat{\Delta}_i \\
\text{for all } i \in I' - J'\text{: } & x_i = 1 - \frac{v + C_i^0}{\overline{L}_i^0} \text{ and } y_i = \widehat{\Delta}_i
\end{aligned}$$

We now show that $(\mathbf{x}, y)$ also satisfies the constraints when using I with the properties stated in the claim. For that, it needs to satisfy the same expressions as above, but with I' and J' replaced by I and J, respectively.

The first condition holds because $I' \subset I$. The second condition holds for all $i \in J - I'$, because $i \notin I'$ satisfies $x_i = 0$ and $0 \leq y_i = 0 \leq \widehat{\Delta}_i$. It also holds for all $i \in J \cap I'$ because $i \in J$ implies $\overline{M}_i^0 = v$, and because $i \in I'$ and

$i \in J'$. For the third condition, note that $I - J \subset I' - J'$ because $i \in I - J$ implies the inequality $\overline{M}_i^0 > v = \max_{k \notin I'} \overline{M}_k^0$; hence, the first inequality in the last expression implies $i \notin J'$, while the equality implies $i \in I'$. $\square$

Proposition 2 stated in the main text follows by combining Propositions B.2, B.3 and B.6.