

Stanton Graph Decompositions

Hau Chan and Dinesh G. Sarvate

ABSTRACT. Stanton graphs S_k (in honor of professor Ralph G. Stanton) are defined, and a new graph decomposition problem for Stanton graphs is proposed. Such decompositions of λK_v for all v 's with minimum λ 's have been obtained for S_3 .

1. Introduction

Let $T = (V, E)$ be a graph with vertex set V and edge set E . A classical problem in combinatorics is to find a decomposition of T into isomorphic copies of a graph, say G . In other words, the problem is to find a G -decomposition of the graph T . In such a decomposition, we can impose further conditions on vertices or on edges. The construction of combinatorial designs can be considered as a decomposition problem, where the pairs of points (edges) meet certain requirements.

For example, consider λ copies of a complete graph K_v of order v (or λK_v). The question of decomposing λK_v into copies of K_k for some k is equivalent to constructing a BIBD(v, k, λ).

In 2007, a new type of design called *strict SB designs* were discussed in [12, 13] and [14].

DEFINITION 1. A *Sarvate-Beam design* $SB(v, k)$ consists of a v -set V and a collection of k -subsets (called *blocks*) of V such that each distinct pair of elements in V occurs with different frequencies; i.e., different pairs occur in a different number of blocks. A *strict* $SB(v, k)$ design is a SB -design where for every i , $1 \leq i \leq \binom{v}{2}$, exactly one pair occurs i times.

EXAMPLE 1. A *strict* $SB(4, 3)$ on $\{1, 2, 3, 4\}$ can be given by the following blocks:

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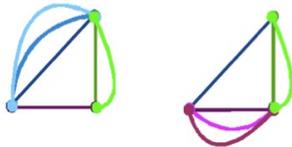
$$\{1, 2, 4\}, \{1, 3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{2, 3, 4\}, \{2, 3, 4\}, \{2, 3, 4\}$$

Although the general existence question of strict SB–designs is still an open question, it has been proven that the necessary conditions are sufficient for $k = 3$ by Dukes [8] (except for some finite number of exceptions). On the other hand, Ma, Chang and Feng [11] have proven that the necessary conditions are sufficient for $k = 3$. Furthermore, Hein and Li provided results on the number of Sarvate–Beam triple systems for $v = 5$ and $v = 6$ [10], and Bradford, Hein, and Pace provided results on Sarvate–Beam quad systems for $v = 6$ [4] to answer some of the questions raised by Stanton [15, 16, 17, 18]. Dukes, Hurd and Sarvate studied SB matrices [9], and Chan and Sarvate studied large sets for SB designs for $k = 2$ as well as 1–SB designs (see [5] and [6]).

Conversely, questions can be asked: Is it possible to decompose (for some minimum number of) copies of a complete graph into graphs on k vertices, where for each $i = 1$ to $\binom{k}{2}$, there is exactly one edge of multiplicity i ? If so, how? In honor of Professor Ralph G. Stanton, we call these graphs *Stanton graphs*, denoted by S_k . Formally,

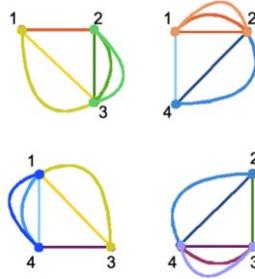
DEFINITION 2. *A Stanton graph of order k , S_k , is a graph on k vertices where for each $i = 1$ to $\binom{k}{2}$, there is exactly one edge of multiplicity i .*

EXAMPLE 2. *Let $\lambda = 4$, $v = 3$, and $k = 3$. We can decompose $4K_3$ into 2 S_3 's*



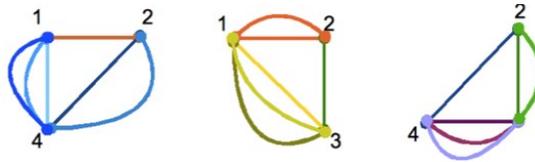
Note that the above example uses 4 copies of K_3 's and we can not decompose a smaller number of copies of K_3 's into Stanton graphs.

EXAMPLE 3. *Given $\lambda = 4$, $v = 4$, and $k = 3$, we can decompose $4K_4$ into 4 S_3 's as follows*



The above example does not use the minimum multiple copies of K_v . The decomposition can be done using a smaller value of λ .

EXAMPLE 4. Consider $\lambda = 3$, $v = 4$, and $k = 3$. We can decompose $3K_4$ into 3 S_3 's as follows



EXAMPLE 5. The decomposition solution of $3K_5$ is given by the triangles:

$$\langle 1, 2, 3 \rangle, \langle 2, 5, 1 \rangle, \langle 3, 1, 4 \rangle, \langle 4, 3, 5 \rangle \text{ and } \langle 5, 4, 2 \rangle$$

where $\langle a, b, c \rangle$ denotes a graph on three vertices $\{a, b, c\}$ with one edge between a and b , two edges between a and c , and three edges between b and c . This notation will be used throughout this note.

In this note we affirmatively answer the question:

“Can we decompose λ copies of K_v (for the minimum λ) into Stanton graphs of k vertices?”

for $k = 3$ after finding the minimal values of λ for a given v .

We need some basic definitions and well-known results from design theory; for example, see [1, 2, 7].

DEFINITION 3. A *Balanced Incomplete Block Design* $BIBD(v, k, \lambda)$ is a collection of k -subsets (called blocks) of a v -set such that each pair of distinct points occurs in exactly λ blocks (where $k < v$).

The definition of a BIBD(v, k, λ) requires $k < v$, but sometimes the notation BIBD(v, v, λ) is used to denote λ copies of the complete block $\{1, 2, \dots, v\}$.

DEFINITION 4. *A parallel class (or a resolution class) in a design is a set of blocks that partitions the point set.*

DEFINITION 5. *A resolvable balanced incomplete block design RBIBD(v, k, λ) is a BIBD(v, k, λ) whose blocks can be partitioned into parallel classes.*

THEOREM 1. *Necessary conditions for the existence of a RBIBD(v, k, λ) are $\lambda(v-1) \equiv 0 \pmod{(k-1)}$ and $v \equiv 0 \pmod{k}$.*

THEOREM 2. *There exists a RBIBD($v, 3, 1$) if and only if $v \equiv 3 \pmod{6}$.*

Let $B = \{b_1, \dots, b_k\}$ be a subset of an additive group G . The list of differences from B is the multiset $\Delta B = \{b_i - b_j \mid i, j = 1, \dots, k; i \neq j\}$.

DEFINITION 6. *Let G be a group of order v . A collection $\{B_1, \dots, B_t\}$ of k -subsets of G form a (v, k, λ) difference family (or difference system) if every nonidentity element of G occurs λ times in the multiset $\Delta B_1 \cup \dots \cup \Delta B_t$. The sets B_i are called base blocks.*

THEOREM 3. *There exists a $(v, 3, 1)$ difference family for every $v \equiv 1, 3 \pmod{6}$.*

For ease of reference, we state Agrawal's theorem [3] and its associated lemma:

LEMMA 1. *Given positive integers v, b, r and k such that $bk = vr$, $v > k$ and a set V of v points, there exists a collection of k -subsets of V such that each point of V is in exactly r subsets: such a collection is called an equi-replicate binary incomplete block design.*

THEOREM 4. *Given any binary equi-replicate design of constant block size k with $bk = vr$ and $b = mv$, the treatments can be rearranged into blocks written as columns, such that every treatment occurs in each row m times.*

2. Minimum Multiplicity for $k = 3$

LEMMA 2. *The graph λK_v can be S_k -decomposed only if $\frac{\binom{k}{2}(\binom{k}{2}+1)}{2}$ divides $\lambda \binom{v}{2}$.*

PROOF. In λK_v , there are a total of $\lambda \binom{v}{2}$ edges. As the graph S_k has $\frac{\binom{k}{2}(\binom{k}{2}+1)}{2}$ edges. The result follows immediately. \square

In particular, for $k = 3$, we have:

COROLLARY 1. *The graph λK_v can be S_3 -decomposed only if 12 divides $\lambda v(v-1)$.*

From Corollary 1, we have

THEOREM 5. *The minimum λ for*

- $v \equiv 2, 11 \pmod{12}$ is 6,
- $v \equiv 3, 6, 7, 10 \pmod{12}$ is 4,
- $v \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$ is 3.

In the next section we prove that for all pairs of v 's and minimum λ 's, S_3 -decompositions of λK_v exist for all $v \geq 3$.

3. S_3 -Decompositions

In this section, we assume that $v \geq 3$.

Construction 1: Let \mathbf{B} be the collection of blocks of a BIBD($v, 3, 1$). For each block $\{a, b, c\}$ in \mathbf{B} , construct two S_3 's: $\langle a, b, c \rangle$ and $\langle c, b, a \rangle$. Hence

THEOREM 6. *If a BIBD($v, 3, 1$) exists, then a S_3 -decomposition exists for $4K_v$.*

In particular, since a BIBD($v, 3, 1$) exists for $v \equiv 1, 3 \pmod{6}$, we have

COROLLARY 2. *A S_3 -decomposition of λK_v exists with minimum $\lambda = 4$ for $v \equiv 3, 7 \pmod{12}$.*

Construction 2: Let \mathbf{B} be the collection of blocks of a BIBD($v, 3, 3$). Each block $\{a, b, c\}$ gives a set of three pairs $\{\{a, b\}, \{b, c\}, \{a, c\}\}$; therefore the collection of all blocks can be considered as a 1-design of $\binom{v}{2}$ pairs with $r = 3$. Therefore by Agrawal's theorem, this 1-design can be arranged so that each pair occur at first, second, and third location. Each block of \mathbf{B} gives a S_3 , where the multiplicity of an edge is the position of that edge in the corresponding set of 1-design of pairs. Hence we have

THEOREM 7. *A S_3 -decomposition of $6K_v$ exists for odd v 's.*

COROLLARY 3. *A S_3 -decomposition of λK_v exists with minimum $\lambda = 6$ for $v \equiv 11 \pmod{12}$.*

Construction 3: Let \mathbf{B} be the collection of blocks of a BIBD($v, 4, 1$). For each block $\{a, b, c, d\}$ in \mathbf{B} , use the S_3 -decomposition of $3K_4$ with vertices $\{a, b, c, d\}$ from example 4.

We have

THEOREM 8. *If a BIBD($v, 4, 1$) exists, then a S_3 -decomposition exists for $3K_v$.*

As BIBD($v, 4, 1$) exists for $v \equiv 0, 1 \pmod{12}$, we have

COROLLARY 4. *An S_3 -decomposition of λK_v exists with minimum $\lambda = 3$ for $v \equiv 0, 1 \pmod{12}$.*

Suppose $v \equiv 1 \pmod{4}$ and $v = 4t + 1$ for some integer $t \geq 1$. Consider a family of t sets: $\{\{1, 2, t+3\}, \{1, 3, t+5\}, \dots, \{1, t+1, 3t+1\}\}$. Note that in this family the differences $1, \dots, t$ occur once as the differences between the first and the second elements of difference sets, and differences $t+1, \dots, 2t$ occur twice: first they occur as the differences between the second and the third elements of the difference sets and then again they occur as the differences between the first and the second elements of difference sets as $t+2, \dots, 2t+1 = 2t, 2t-2, \dots, 3t = t+1$.

Construction 4: Let \mathbf{B} be the collection of ordered blocks generated by the family of t sets: $\{\{1, 2, t+3\}, \{1, 3, t+5\}, \dots, \{1, t+1, 3t+1\}\}$. For each block $\{a, b, c\}$ in \mathbf{B} , construct $S_3 : \langle a, c, b \rangle$ to obtain the following theorem.

THEOREM 9. *For $v \equiv 1 \pmod{4}$, a S_3 -decomposition exists for $3K_v$.*

COROLLARY 5. *A S_3 -decomposition of λK_v exists with minimum $\lambda = 3$ for $v \equiv 5, 9 \pmod{12}$.*

Suppose $v \equiv 0 \pmod{4}$ and let $v = 4(t+1)$ for some integer $t \geq 1$. Consider a family of t sets: $\{\{1, 2, t+4\}, \{1, 3, t+6\}, \dots, \{1, t+1, 3t+2\}\}$. Note that in this family the differences $1, \dots, t$ occur once as the differences between the first and the second elements of difference sets, differences $t+2, \dots, 2t+1$ occur twice: first they occur as the differences between the second and the third elements of the difference sets and then again they occur as the differences between the first and the third elements of the difference sets as $t+3, t+5, \dots, 2t+1 = 2t+2, \dots, t+2$, whereas the difference of $t+1$ does not appear anywhere.

Construction 5: Let \mathbf{B} be the collection of blocks generated by the family of t sets: $\{\{1, 2, t+4\}, \{1, 3, t+6\}, \dots, \{1, t+1, 3t+2\}\}$ on $4t-1$ elements and let \mathbf{C} be the collection of blocks generated by the difference family $\{\infty, 1, t+2\}$ on $4t-1$ elements. For each block $\{a, b, c\}$ in \mathbf{B} , construct $S_3 : \langle a, c, b \rangle$. For each block $\{\infty, a, b\}$ in \mathbf{C} , construct $S_3 : \langle a, \infty, b \rangle$. We are using the common convention while generating blocks from a difference set containing the element ∞ , the infinity element remains fixed while other elements cycle through mod $4t-1$ to give $4t-1$ distinct blocks.

The S_3 's constructed above give a S_3 -decomposition. Hence along with Example 4 we have:

THEOREM 10. *For $v \equiv 0 \pmod{4}$, a S_3 -decomposition exists for $3K_v$.*

COROLLARY 6. *A S_3 -decomposition of λK_v exists with minimum $\lambda = 3$ for $v \equiv 0, 4, 8 \pmod{12}$.*

Construction 6: Suppose $v \equiv 2 \pmod{12}$ and let $v = 12t + 2$ for some integer $t \geq 1$. Since $12t + 1 \equiv 1 \pmod{6}$, a difference family with $(t + 1)$ base blocks exists for BIBD(12t + 1, 3, 1) (by Theorem 3). Let \mathbf{B} be the collection of ordered blocks generated by any t base blocks. For each block $\{a, b, c\}$ in \mathbf{B} , construct three S_3 's: $\langle a, b, c \rangle$, $\langle c, a, b \rangle$, and $\langle b, c, a \rangle$. Replace block $\{a, b, c\}$ generated by the remaining base block by four S_3 's with a new point ∞ : $\langle b, c, a \rangle$, $\langle a, c, b \rangle$, $\langle b, \infty, c \rangle$, and $\langle c, \infty, a \rangle$.

This construction gives:

THEOREM 11. *For $v \equiv 1 \pmod{6}$ ($v > 6$), a S_3 -decomposition exists for $6K_{v+1}$.*

COROLLARY 7. *A S_3 -decomposition of λK_v exists with minimum $\lambda = 6$ for $v \equiv 2 \pmod{12}$.*

Construction 7: Suppose $v \equiv 6 \pmod{12}$ and let $v = 12t + 6$ for some integer $t \geq 1$. Since $12t + 3 \equiv 3 \pmod{6}$, a resolvable BIBD($v - 3, 3, 1$) exists. Note that there are at least $r \geq 4$ parallel classes. Let \mathbf{B}_i be the collection of blocks given by the first three parallel classes $P_i, i = 1, 2, 3$, respectively. Let \mathbf{C} be the collection of blocks given by the parallel classes P_j where $j = 4, \dots, r$. For each block $\{a, b, c\}$ in \mathbf{B}_i , construct S_3 : $\langle a, b, c \rangle$ and also using the same block and ∞_i , construct three S_3 's: $\langle b, \infty_i, a \rangle$, $\langle \infty_i, c, b \rangle$, and $\langle c, a, \infty_i \rangle$. For each block $\{a, b, c\}$ in \mathbf{C} , construct two S_3 's: $\langle a, b, c \rangle$ and $\langle a, c, b \rangle$. Finally decompose $4K_3$ with vertices $\{\infty_1, \infty_2, \infty_3\}$ as in Example 2.

These S_3 's we constructed give a S_3 -decomposition of $4K_{12t+6}$, and we have:

THEOREM 12. *If a resolvable BIBD($v, 3, 1$) exists, then a S_3 -decomposition exists for $4K_{v+3}$.*

EXAMPLE 6. *The decomposition of $4K_6$ is given by the triangles*

$$\begin{aligned} &\langle 3, 1, 2 \rangle, \langle 2, 1, 4 \rangle, \langle 4, 1, 5 \rangle, \langle 5, 1, 6 \rangle, \langle 6, 1, 3 \rangle, \\ &\langle 4, 6, 2 \rangle, \langle 2, 5, 3 \rangle, \langle 5, 3, 4 \rangle, \langle 6, 2, 5 \rangle, \text{ and } \langle 3, 4, 6 \rangle \end{aligned}$$

Along with example 6, we have

COROLLARY 8. *A S_3 -decomposition of λK_v exists with minimum $\lambda = 4$ for $v \equiv 6 \pmod{12}$.*

Construction 8: Suppose $v \equiv 10 \pmod{12}$ and let $v = 12t + 10$ for some integer $t \geq 1$. Since $12t + 9 \equiv 3 \pmod{6}$, a resolvable BIBD($v - 1, 3, 1$) exists. Let \mathbf{B} be the collection of blocks given by the first parallel class P_1 . Let \mathbf{C} be the collection of blocks given by the remaining parallel classes. For each block $\{a, b, c\}$ in \mathbf{B} , construct S_3 : $\langle a, b, c \rangle$ and using the same block with ∞_1 , construct three S_3 's: $\langle b, \infty_1, a \rangle$, $\langle \infty_1, c, b \rangle$, and

$\langle c, a, \infty_1 \rangle$. For each block $\{a, b, c\}$ in \mathbf{C} , construct two S_3 's: $\langle a, b, c \rangle$ and $\langle a, c, b \rangle$.

We have

THEOREM 13. *If a resolvable BIBD($v, 3, 1$) exists, then a S_3 -decomposition exists for $4K_{v+1}$.*

COROLLARY 9. *A S_3 -decomposition of λK_v exists with minimum $\lambda = 4$ for $v \equiv 10 \pmod{12}$.*

4. Summary

In the above section, we have shown that for all v 's ≥ 3 , the minimum copies of K_v 's can be S_3 -decomposed. Below is a table that summarizes the results:

TABLE 1. Results

v	minimum λ	Construction
0, 1 (mod 12)	3	Corollary 4
2 (mod 12)	6	Corollary 7
3, 7 (mod 12)	4	Corollary 2
4, 8 (mod 12)	3	Corollary 6
5, 9 (mod 12)	3	Corollary 5
6 (mod 12)	4	Corollary 8
10 (mod 12)	4	Corollary 9
11 (mod 12)	6	Corollary 3

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(A. One) STONY BROOK UNIVERSITY, DEPT. OF COMPUTER SCIENCE, STONY BROOK, NY, 11790

E-mail address: `hauchan@cs.stonybrook.edu`

(A. Two) COLLEGE OF CHARLESTON, DEPT. OF MATH., CHARLESTON, SC, 29424

E-mail address: `sarvated@cofc.edu`