A non-existence result and large sets for Sarvate-Beam designs

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ABSTRACT. It is shown that for $2 \leq t \leq n-3$, a strict t-SB$(n, n-1)$ design does not exist, but for $n \geq 3$, a non-strict 2-SB$(n, n-1)$ design exists. The concept of large sets for Steiner triple systems is extended to SB designs and examples of a large sets for SB designs are given.

1. Introduction

Stanton [**9**] renamed a type of block design that was introduced in [**7**] as Sarvate-Beam Triple Systems (SB Triple Systems). In addition, Stanton obtained several interesting results and raised questions on enumeration and existence, see [**10**], [**11**], [**12**] and [**13**]. Some of these questions are solved by Hein and Li [**5**] as well as Bradford, Hein and Pace [**1**]. In general, an SB design is a block design in which every pair occurs in a different number of blocks. Below is a formal definition:

DEFINITION 1. *A Sarvate-Beam design, SB(v,k), consists of a v-set V and a collection of k-subsets, called blocks, of V such that each distinct pair of elements in V occurs with different frequencies i.e., in a different number of blocks. A strict SB(v,k) design is a design where for every i, $1 \leq i \leq \binom{v}{2}$, exactly one pair occurs exactly i times.*

EXAMPLE 1. *A strict SB$(4,3)$ on $\{1,2,3,4\}$ consists of the following blocks:*

$$\{1,2,4\},\ \{1,3,4\},\ \{1,3,4\},\ \{2,3,4\},\ \{2,3,4\},\ \{2,3,4\},\ \{2,3,4\}.$$

Although the general existence question of strict SB block designs is still an open question, it has been proven that the necessary conditions are

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sufficient for $k = 3$ by Dukes [**3**] except for some finite number of exceptions. On the other hand, Ma, Chang and Feng [**6**] have proved that the necessary conditions are sufficient for $k = 3$. Moreover, SB matrices have been studied by Dukes, Hurd and Sarvate [**4**]. The following definition and result appear in [**8**]:

DEFINITION 2. *A t-SB(v,k) design is a collection, B, of k-subsets of a v-set such that each t-subset of V occurs a distinct number of times. In a strict t-SB design, for each i,* $1 \leq i \leq \binom{v}{t}$*, there is exactly one t-subset which occurs in i blocks.*

THEOREM 1. *A strict t-SB(v,k) exists only if*

$$\binom{k}{t} \mid \frac{\binom{v}{t}(\binom{v}{t}+1)}{2}.$$

2. Non-existence result

The following result is known [**8**]:

THEOREM 2. *For* $n > 4$*, a strict* $(n-2)$*-SB*$(n, n-1)$ *does not exist.*

We prove the following result:

THEOREM 3. *For* $n > 4$*, a strict t-SB*$(n, n-1)$ *does not exist for* $2 \leq t \leq n-3$*.*

PROOF. Let us denote the frequency of an s-subset, $\{a_1, a_2, ..., a_s\}$, in the design by $f(a_1, ..., a_s)$. Let $B_i = \{1, 2, ..., n\} - \{i\}$, $i = 1, 2, \cdots, n$, be the n subsets of size $n-1$ of $\{1, 2, \cdots, n\}$. Let $F(B_i)$ denotes the frequency of the block B_i in the design if it exists. Without loss of generality, assume that the t-subset {1, 2, .., t} appears exactly once and let $B_n = \{1, 2, ..., t, ..., n-1\}$ be the block containing $\{1, 2, \cdots, t\}$ that appears exactly once. Observe that there are $n-t$ sets, $B_{t+1}, B_{t+2}, ..., B_n$, among $B_1, B_2, ..., B_{n-1}, B_n$ containing $\{1, 2, ..., t\}$, and $n-t+1$ sets, $B_t, B_{t+1}, ..., B_n$, containing $\{1, 2, ..., t-1\}$. As the frequency of $\{1, 2, ..., t\}$ is one and $F(B_n) = 1$, it follows that $F(B_{t+1}) = F(B_{t+2}) = ... = F(B_{n-1}) = 0$. Hence, there exists only one other set, B_t, which contains $\{1, 2, ..., t-1\}$ but not $\{1, 2, ..., t\}$ whose frequency (say ϕ) may be greater than one in the design. This is the only set other than B_n which contains $\{1, 2, ..., t-1, x\}$ and $\{1, 2, ..., t-1, y\}$, where $x, y \in \{t, ..., n\}$ and $x \neq y$. Hence $f(1, 2, ..., t-1, x) = \phi + 1 = f(1, 2, ..., t-1, y)$, which is a contradiction. □

The following example is illustrative:

EXAMPLE 2. *A strict* 3*-SB*(6,5) *does not exist. First note that the design parameters satisfy Theorem 1. There are* 6 *subsets* $\{1, 2, 3, 4, 5\}$*,* $\{1, 2, 3, 4, 6\}$*,* $\{1, 2, 3, 5, 6\}$*,* $\{1, 2, 4, 5, 6\}$*,* $\{1, 3, 4, 5, 6\}$*,* $\{2, 3, 4, 5, 6\}$*. Without loss of generality, assume the* 3*-subset* $\{1, 2, 3\}$ *occurs exactly once in* *the block* $\{1,2,3,4,5\}$. *Note that we cannot have blocks* $\{1,2,3,4,6\}$ *and* $\{1,2,3,5,6\}$ *in this design since we want* $\{1,2,3\}$ *to appear exactly once. Therefore the remaining blocks must be some multiple copies of the sets* $\{1,2,4,5,6\}$, $\{1,3,4,5,6\}$, *and* $\{2,3,4,5,6\}$.

Let a, b, *and* c *denote the frequency of the blocks* $\{1,2,4,5,6\}$, $\{1,3,4,5,6\}$, *and* $\{2,3,4,5,6\}$ *respectively, if the design exists. Note* $f(1,2,4) = 1 + a = f(1,2,5)$, *which is a contradiction.*

3. Non-strict 2-SB$(n, n-1)$ designs

Although strict 2-SB$(n, n-1)$ designs do not exist for any n, non-strict 2-SB$(n, n-1)$ designs exist for all $n \geq 3$:

Lemma 1. *A non-strict* t*-SB*$(n, n-1)$ *design is also a non-strict* $(t-1)$*-SB*$(n, n-1)$ *design if* $n - 1 \geq 2t - 2$.

Proof. Suppose the block $B_i = \{1, 2, \cdots, n\} - \{i\}$ occurs f_i times in the non-strict t-SB$(n, n-1)$ design. A $(t-1)$-set $\{i_1, i_2, \cdots, i_{t-1}\}$ occurs in b-$(f_{i_1} + f_{i_2} + \cdots + f_{i_{t-1}})$ blocks, where b is the total number of blocks of the non-strict t-SB$(n, n-1)$ design. If the design is not a non-strict $(t-1)$-SB$(n, n-1)$ design, then there exists at least two distinct $(t-1)$-sets, $\{a_1, a_2, \cdots, a_{t-1}\}$ and $\{b_1, b_2, \cdots, b_{t-1}\}$ both occurring the same number of times (say μ) in the design. As $2t - 2 \leq n - 1$, there exists an element a in $\{1, 2, \cdots, n\}$ but not in the union of $\{a_1, a_2, \cdots, a_{t-1}\}$ and $\{b_1, b_2, \cdots, b_{t-1}\}$. Consider the t-sets $\{a, a_1, a_2, \cdots, a_{t-1}\}$ and $\{a, b_1, b_2, \cdots, b_{t-1}\}$. Clearly both occur in $\mu - f_a$ blocks of the non-strict t-SB$(n, n-1)$ design which is a contradiction. □

In general a t-SB(n, k) design need not be a $(t-1)$-SB(n, k) design as shown below:

Example 3. *Let* $V = \{1, 2, 3, 4\}$. *The collection of blocks with* t *copies of* $\{1, 2\}$, *one copy of* $\{1, 3\}$, *four copies of* $\{1, 4\}$, *two copies of* $\{2, 3\}$, *three copies of* $\{2, 4\}$ *and* s *copies of* $\{3, 4\}$ *for any distinct values of* s *and* t *different from* $1, 2, 3$, *and* 4 *provides a* 2*-SB*$(4, 2)$ *design. The design is a strict* 2*-SB*$(4, 2)$ *when* $\{s, t\}$=$\{5, 6\}$. *Note that the elements* 1 *and* 2 *both have the replication number* $t + 5$ *and hence the design is not a* 1*-SB*$(4, 2)$ *design.*

Theorem 4. *A non-strict* 2*-SB*$(n, n-1)$ *design exists for every positive integer* $n \geq 3$.

Proof. Let the set of elements be $\{1, 2, \cdots, n\}$ for a design on n elements. The proof is based on induction. For $n = 3$, a non-strict 2-SB$(3, 2)$ design can be easily constructed. Suppose we have a non-strict 2-SB$(n, n-1)$ for some value of n with b blocks. We construct a 2-SB$(n+1, n)$ design containing $2b$ blocks using the blocks of the non-strict 2-SB$(n, n-1)$

design and the set $\{1, 2, \cdots, n\}$ as follows. First we construct b blocks by adding the element $n+1$ into each block of the non-strict 2-SB$(n, n-1)$ design. Note that in these blocks each element from 1 to n occurs different number of times, therefore the pairs $\{n+1, i\}$ occur different number of times. We complete the construction of non-strict 2-SB$(n+1, n)$ design by including b copies of the set $\{1, 2, \cdots, n\}$. The maximum number of times a pair $\{n+1, i\}$ may have occurred is b, and minimum number of times a pair $\{i, j\}$, $1 \leq i < j \leq n$, occurs in the non-strict 2-SB$(n, n-1)$ design is one. Hence, all pairs occur a different number of times in these $2b$ blocks of the non-strict 2-SB$(n+1, n)$ design. □

4. Large sets

Definition 3. *A triple system $(V, \boldsymbol{B})$ is a set V of v elements together with a collection $\boldsymbol{B}$ of 3-subsets (called blocks or triples) of V with the property that every 2-subset of V occurs in exactly λ blocks. The size of V is the order of the triple system. It is also denoted by $TS(v, \lambda)$, or Steiner triple system, STS(v), when $\lambda = 1$.*

Definition 4. *Let (V,B) and (V,D) be two $STS(v)$'s. Their intersection size is $|B \cap D|$. They are disjoint when their intersection size is zero. A set of $(v-2)$ $STS(v)$s, $\{(V,B_i) : i=1,...,v\text{-}2\}$, is a large set if any two distinct systems from the set are disjoint.*

In other words, the set of all 3-subsets of a v-set is partitioned into v-2 STS(v)'s. It is known that large sets for triple systems exist for all $v \equiv$ 1,3(mod 6) except for $v = 7$ [**2**].

The analogous question to the large set for triple system with respect to SB triple systems can be formulated using the following definition:

Definition 5. *Let V be a v-set. A family of $SB(v,k)$ designs on V, say B=$\{B_1, B_2, ..., B_n\}$, is a large set with multiplicity s if $\dot{\bigcup}_{i=1}^{n} B_i$ gives s copies of the set of all k-subsets of V for some integer s and if there is another family of $SB(v,k)$ designs C=$\{C_1, C_2, ..., C_m\}$ where $\dot{\bigcup}_{i=1}^{m} C_i$ contains t copies of all k-subsets of V, then $s \leq t$.*

Simple counting gives the following result:

Theorem 5. *Suppose the multiplicity for the large set for a $SB(v,k)$ is s and let the size of the large set be n. Then $s\binom{v}{k} = \frac{\binom{v}{2}(\binom{v}{2}+1)}{2\binom{k}{2}} \times n$; hence a necessary condition for the existence of a large set for strict $SB(v,k)$ is $\frac{\binom{v}{2}(\binom{v}{2}+1)}{2\binom{k}{2}} \mid s\binom{v}{k}$.*

Corollary 1. *For $k = 3$, $\frac{\binom{v}{2}(\binom{v}{2}+1)}{6} \mid s\binom{v}{3}$.*

The following example will clarify the definition:

Example 4. *Consider the set $V = \{1, 2, 3, 4\}$. We have the following 4 strict $SB(4, 3)$'s.*

- $B_1 = \{\{1,2,4\}, \{1,3,4\}, \{1,3,4\}, \{2,3,4\}, \{2,3,4\}, \{2,3,4\}, \{2,3,4\}\}$
- $B_2 = \{\{1,2,3\}, \{1,2,3\}, \{1,2,3\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}, \{2,3,4\}\}$
- $B_3 = \{\{1,2,3\}, \{1,2,3\}, \{1,2,4\}, \{1,2,4\}, \{1,2,4\}, \{1,2,4\}, \{2,3,4\}\}$
- $B_4 = \{\{1,2,3\}, \{1,2,4\}, \{1,2,4\}, \{1,3,4\}, \{1,3,4\}, \{1,3,4\}, \{1,3,4\}\}$.

When we take the multi-union $B_1 \dot\cup B_2 \dot\cup B_3 \dot\cup B_4$, we get a multi-set where each of the blocks $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, and $\{2, 3, 4\}$ occurs 7 times. Indeed 7 is the multiplicity for $SB(4, 3)$, because there are only 4 distinct blocks and $SB(4, 3)$ has 7 blocks, if the multiplicity is s, then $4 \times s = 7 \times n$ for some integer n. Therefore $\{B_1, B_2, B_3, B_4\}$ is the large set for $SB(4, 3)$, and as the $SB(4, 3)$ is unique, the large set is unique up to isomorphism.

Example 5. *A set of $SB(6, 3)$ designs such that the multi-union of the collections of blocks has multiplicity $t = 10$ is given below, however this may not be a large set. The reason is that we obtained these designs by taking isomorphic copies of a single $SB(6, 3)$ design, but according to [**5**], there are 48, 843 non-isomorphic restricted $SB(6, 3)$, and a total of 16, 444, 250 (restricted and non-restricted) $SB(6, 3)$ designs. What we can claim is that using this particular design, the multiplicity cannot be less than 5.*

Blocks	Design 1	Design 2	Design 3	Design 4	Design 5
{1,2,3}	0	0	2	4	4
{1,2,4}	0	1	1	5	3
{1,2,5}	1	0	3	1	5
{1,2,6}	0	2	3	3	2
{1,3,4}	0	4	0	4	2
{1,3,5}	1	1	2	1	5
{1,3,6}	1	5	1	2	1
{1,4,5}	2	3	0	2	3
{1,4,6}	1	5	0	3	2
{1,5,6}	2	3	3	0	2
{2,3,4}	1	1	1	5	2
{2,3,5}	2	0	4	0	4
{2,3,6}	2	1	5	1	1
{2,4,5}	3	0	2	2	3
{2,4,6}	3	2	2	3	0
{2,5,6}	3	1	5	0	1
{3,4,5}	5	2	1	1	1
{3,4,6}	4	4	0	2	0
{3,5,6}	4	2	4	0	0
{4,5,6}	5	3	1	1	1

4.1. Large sets for $k = 2$. Let us consider the following two examples:

EXAMPLE 6. *A strict SB*$(3, 2)$ *design with blocks* $\{\{1, 2\}, \{1, 3\}, \{1, 3\}, \{2, 3\}, \{2, 3\}, \{2, 3\}\}$.

EXAMPLE 7. *Another strict SB*$(3, 2)$ *design with blocks* $\{\{1, 2\}, \{1, 2\}, \{1, 2\}, \{1, 3\}, \{1, 3\}, \{2, 3\}\}$.

The union of these designs is a multi-set that contains each 2-subset with a multiplicity of 4. In fact, these designs form a large set. This simple observation leads to the following result:

THEOREM 6. *Large sets with multiplicity* $\binom{v}{2}+1$ *containing exactly two* $SB(v, 2)'s$ *exist for all* $v \geq 2$.

PROOF. Let the 2-subsets of a v-set V be $\{b_1, b_2, ..., b_{\binom{v}{2}}\}$. Without loss of generality, let the first SB$(v, 2)$, B_1, contain blocks b_i with frequency i. Now construct a second SB$(v, 2)$, B_2, where b_i occurs with frequency $\binom{v}{2}$ +1- i. It follows that we have a partition $\{B_1,\ B_2\}$ of the collection of the 2-subsets of V with multiplicity $\binom{v}{2}+1$. □

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