Beautifully Ordered Balanced Incomplete Block Designs

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ABSTRACT. Beautifully Ordered Balanced Incomplete Block Designs, BOBIBD($v, k, \lambda, k_1, \lambda_1$), are defined and the proof is given to show that necessary conditions are sufficient for the existence of BOBIBD with block size k=3 and k_1=2 and for k=4 and k_1=2 except possibly for eleven exceptions. Existence of BOBIBDs with block size k=4 and k_1=3 is demonstrated for all but one infinite family and the non-existence of BOBIBD($7, 4, 2, 3, 1$), the first member of the series, is shown.

1. Introduction

A *Balanced Incomplete Block design*, BIBD(v, k, λ), is a collection of k-subsets (called blocks) of a v-set such that each pair of distinct points occurs in exactly λ blocks where $k<v$. A *Nested Balanced Incomplete Block Design*, (NBIBD), is a BIBD(v, k, λ) in which it is possible to subdivide each block of the design into $\frac{k}{k_1}$ sub-blocks of size k_1 such that the sub-blocks themselves form a BIBD, here k and k_1 are positive integers such that k_1 divides k. For example, consider the following collection of five blocks of a BIBD(5,4,3) on five points {1,2,3,4,5}:

{{1,2,3,4}, {1,2,3,5}, {1,2,4,5}, {1,3,4,5}, {2,3,4,5}}.

Now consider the following subdivision of these five blocks in two specific blocks of size two:

block {1,2,3,4} into blocks {1,4}, and {2,3},
block {1,2,3,5} into blocks {3,5}, and {1,2},
block {1,2,4,5} into blocks {4,2}, and {1,5},
block {1,3,4,5} into blocks {3,1}, and {4,5},

2000 *Mathematics Subject Classification.* Primary 05B05.

Key words and phrases. Block designs, BIBDs, Nested designs, ordered designs, BOBIBD.

block {2,3,4,5} into blocks {2,5}, and {3,4}.

Notice that these ten subsets of size two form a BIBD(5,2,1) and hence above BIBD(5,4,3) is an NBIBD.

In the above example one can not arbitrarily partition each block of the original design into two blocks to get a BIBD, of course for an NBIBD this condition is not even required. Nested designs have been studied extensively [**8**]. The BIBDs with ordered blocks are also studied extensively in different context, for example see [**7**] and [**2**].

We are proposing to order the elements of the blocks of a BIBD in such a way that for any fixed set of k_1 locations, the collection of sub-blocks with entries from the fixed set of locations from all blocks gives a BIBD. An example will make this idea clearer: let us takes two copies of BIBD$(5,4,3)$ to get a BIBD$(5,4,6)$ and order the elements of the blocks as follows:

{{1,2,3,4},{2,3,4,5},{3,4,5,1},{4,5,1,2},{5,1,2,3},
{1,3,5,2},{3,5,2,4},{5,2,4,1},{2,4,1,3},{4,1,3,5}}.

As the block size is four, each block has four locations, first, second, third and fourth. Choose ANY two locations, say first and fourth, and construct blocks from the entries at these locations of each block:

{{1, 4}, {2, 5}, {3,1}, {4, 2}, {5, 3}, {1, 2}, {3, 4},{5, 1},{2, 3}, {4, 5}}.

As every pair has occurred exactly once in these (unordered) smaller blocks, we get a BIBD$(5,2,1)$. One can choose any other two distinct locations, viz., first and second, first and third, second and third, second and fourth or third and fourth and construct sub-blocks from the entries at these locations of the ordered blocks and one will get a BIBD$(5,2,1)$. We call such a BIBD with ordered blocks a *Beautifully Ordered Balanced Incomplete Block Design.* Formally,

Definition 1. *If each of the blocks of a* BIBD(v,k,λ) *is ordered such that for any* k_1 *indices* $i_1, i_2, \cdots, i_{k_1}$ *the sub-blocks* $\{a_{i_1}, a_{i_2}, \cdots, a_{i_{k_1}}\}$ *of all ordered blocks* $\{a_1, a_2, \cdots, a_k\}$ *of the* BIBD(v,k,λ) *form a* BIBD(v,k_1,λ_1) *then we say that the collection of ordered blocks gives a Beautifully Ordered Balanced Incomplete Block Design,* BOBIBD$(v,k,\lambda,k_1,\lambda_1)$ *where* $2 \leq k_1 \leq k$-1.

Clearly when k_1 divides k, a BOBIBD gives a nested BIBD with (super) block size k and sub-block size k_1 but for a BOBIBD there is no restriction on k_1, hence BOBIBDs can be constructed even when k_1 is not a factor of k. Examples of BOBIBDs may be given as a $b \times k$ array where the rows are the ordered blocks.

The definition a BIBD(v, k, λ) requires $k<v$, but sometimes the notation BIBD(v, v, λ) is used to denote λ copies of the complete block $\{1, 2, \cdots, v\}$.

EXAMPLE 1. *The following is a* BOBIBD$(5, 5, 10, 2, 1)$.

1	3	5	2	4
3	5	2	4	1
5	2	4	1	3
2	4	1	3	5
4	1	3	5	2
1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

There is another combinatorial object which is quite similar to what we have defined. It is called Perpendicular Array [**3**]. The formal definition is:

DEFINITION 2. *A perpendicular array PA_λ (t,k,v) is a $k \times \lambda\binom{v}{t}$ array with v entries such that*

(1) *each column has k distinct entries, and*
(2) *each set of t rows contains each set of t distinct entries as a column precisely λ times.*

Clearly when $t=2$, the perpendicular array gives a BOBIBD with $k_1=2$ when we consider the columns of the array as the blocks of the BIBD(v, k, λ). Of course, for $k_1 \geq 3$, BOBIBD and perpendicular arrays are different combinatorial structures.

There are many existence results on perpendicular arrays as given in [**3**], for example, using our terminology, it is given in [**3**] that:

- BOBIBD$(v, 3, 3, 2, 1)$ exists for $v \geq 3$, [**10**]
- BOBIBD$(v, 4, 6, 2, 1)$ exists for odd $v \geq 5$ [**9**], [**5**] and
- BOBIBD$(v, 4, 12, 2, 2)$ exists for $v \geq 4$ [**9**]

In fact, the results proven in this paper for $k_1=2$ can be deduced from these results, though we give straight-forward independent proofs for general λ with usual design theory techniques. The results for $k_1=3$ may not be obtained from the results in [**3**]. Given such close relation, one may tempt to rewrite our definition and introduce BOBIBD as an array

DEFINITION 3. *A Beautiful Array $BA(v, k, \lambda, k_1, \lambda_1)$ is a $b \times k$ array ($k > 2$), where $b = \frac{\lambda v(v-1)}{k(k-1)} = \frac{\lambda_1 v(v-1)}{k_1(k_1-1)}$, the entries of which are drawn from a set of v symbols and are disposed so that* (a) *the rows of the array constitute the blocks of a* BIBND(v, k, λ)*, and* (b) *if we form a $b \times k_1$ sub-array from any*

k_1 columns of the array, $1<k_1<k$, then the rows of the sub-array constitute the blocks of a BIBD(v, k_1, λ_1).

We feel that having the word BIBD in the definition is more revealing than the word array as the underlying structure is a BIBD and substructures are also BIBDs.

1.1. Latin Square. We will need the following well known results about Latin squares. For basic definition and notation, please see [**12**]. A Latin square L of side n on symbols $Q = \{1, 2, \cdots, n\}$ can be considered as a Quasigroup $(Q, \circ)$, the rows and columns of L are labeled by the symbols in Q and $i \circ j$ is the $(i, j)^{th}$ element of L. When $(i, i)^{th}$ element of L is i for all $i = 1, 2, \cdots, n$, L is called an idempotent Latin square. Let $N(n)$ denote the number of Latin squares in the largest possible set of mutually orthogonal Latin squares of side n.

Lemma 1. *([**12**], page 126) There exists a set of $N(n)$ - 1 mutually orthogonal idempotent Latin squares of side n.*

Theorem 1. *([**12**], page 143) There exist three mutually orthogonal Latin squares of every side except 2, 3, 6, and possibly 10.*

Corollary 1. *([**12**], page 145) There is a pair of orthogonal idempotent Latin squares of every side except 2, 3 and 6.*

2. Necessary Conditions for BOBIBDs

From definition, if a BOBIBD$(v,k,\lambda,k_1,\lambda_1)$ exists, then

(1) BIBD(v,k,λ) exists, and
(2) BIBD(v,k_1,λ_1) exists.

Hence:

Theorem 2. *Every necessary condition for the existence of* BIBD(v,k,λ) *is a necessary condition for* BOBIBD$(v,k,\lambda,k_1,\lambda_1)$ *and every necessary condition for* BIBD(v,k_1,λ_1) *is a necessary condition for* BOBIBD$(v,k,\lambda,k_1,\lambda_1)$.

For ease of reference the well known necessary conditions for BIBD$(v,3,\lambda)$ and BIBD$(v,4,\lambda)$, for $v \geq k$, are given below:

Block size 3:

λ	spectrum of λ-fold triple systems
$\lambda\equiv0 \pmod 6$	all $v\neq2$
$\lambda\equiv1,5 \pmod 6$	all $v\equiv1,3 \pmod 6$
$\lambda\equiv2,4 \pmod 6$	all $v\equiv0,1 \pmod 3$
$\lambda\equiv3 \pmod 6$	all odd v

Block size 4:

λ	spectrum of λ-fold quadruple systems
$\lambda\equiv 0 (\text{mod } 6)$	all v
$\lambda\equiv 1,5 (\text{mod } 6)$	all $v\equiv 1,4 (\text{mod } 12)$
$\lambda\equiv 2,4 (\text{mod } 6)$	all $v\equiv 1 (\text{mod } 3)$
$\lambda\equiv 3 (\text{mod } 6)$	all $v\equiv 0,1 (\text{mod } 4)$

Simple counting arguments give

THEOREM 3. *In a* BOBIBD$(v,k,\lambda,k_1,\lambda_1)$, $\lambda = \frac{\binom{k}{k_1}\lambda_1}{\binom{k-2}{k_1-2}}$.

COROLLARY 2. (1) *For* $k_1=2$, $\lambda=\binom{k}{2}\lambda_1$ *and the number of blocks must be a multiple of* $\binom{v}{2}$.

(2) *For* $k_1=3$, $\lambda=\frac{(\binom{k}{3}\lambda_1)}{(k-2)}$, *hence* $\lambda=\frac{\binom{k}{2}\lambda_1}{3}$.

THEOREM 4. *If a* BOBIBD$(v,k,\lambda,2,\lambda_1)$ *exists then* k *divides* r *and in the* (*ordered*) *blocks of* BOBIBD *each element occurs exactly* $\frac{r}{k}$ *times at each location of the blocks.*

PROOF. Let c_i denote the number of times an element a appears at the i^{th} location in the collection of ordered blocks of a BOBIBD$(v,k,\lambda,2,\lambda_1)$. Consider any two locations i and j, as we have a BOBIBD with $k_1 = 2$, $c_i+c_j=\lambda_1(v-1)$. Similarly for locations i and k, $c_i+c_k=\lambda_1(v-1)$, hence for all $k\neq j$ $c_k = c_j$. As $c_1+c_2+...c_k$=r, $kc_j=r$, and hence k divides r.

□

EXAMPLE 2. BOBIBD(4,4,6,2,1) *does not exist as* r=6 *and* 4 *does not divide* 6.

The above theorem can be generalized easily as follows.

THEOREM 5. *If a* BOBIBD$(v,k,\lambda,k_1,\lambda_1)$ *exists then* k *divides* r *and in the* (*ordered*) *blocks of* BOBIBD *each element occurs exactly* $\frac{r}{k}$ *times at each location of the blocks.*

As we noted earlier, if we want to construct BOBIBD$(v,k,\lambda,2,\lambda_1)$, there are $\binom{k}{2}$ ways we can pick up two locations in a block of BIBD(v,k,λ), hence λ is a multiple of $\binom{k}{2}$ and $\lambda=\binom{k}{2}\lambda_1$.

THEOREM 6. *If a* BOBIBD$(v,k,\lambda,2,\lambda_1)$ *exists, then a* BOBIBD$(v,k,\lambda,$ $k_1,\binom{k_1}{2}\lambda_1)$ *exists for* $2\leq k_1\leq k$.

Note that the converse is NOT true as shown below.

EXAMPLE 3. *A* BOBIBD(4,4,4,3,2) *with blocks* {1,2,3,4}, {4,1,2,3}, {3,4,1,2}, {2,3,4,1} *is not a* BOBIBD(4,4,4,2,1).

In view of the above theorem, all results obtained for k=4 and k_1=2 extend to k=4 and k_1=3 as well and all examples constructed for BOBIBD($v, 4, \lambda, 2, \lambda_1$) are examples for BOBIBD($v, 4, \lambda, 3, 3\lambda_1$).

THEOREM 7. *If a* BOBIBD(v,k,λ,2,λ_1) exists, then a BOBIBD(v,$k^{'}$, $\binom{k^{'}}{2}\lambda_1$,2,$\lambda_1$) exists, where $2 \leq k^{'} \leq k$.

EXAMPLE 4. *The* BOBIBD(5, 5, 10, 2, 1) *given in the introduction is also a* BOBIBD(5, 5, 10, 3, 3).

The above example has an easy generalization:

THEOREM 8. *A* BOBIBD($v, v, \lambda, k_1, \lambda_1$) *is also a* BOBIBD($v, v, \lambda, v$-$k_1, \lambda_2$)*, where* $\lambda_2 = \frac{\lambda(v-k_1)(v-k_1-1)}{v(v-1)}$ *if* v-$k_1 \geq 2$.

PROOF. As we know that entries from any k_1 locations is a BIBD(v, k_1, λ_1), and hence the compliments of the blocks is a BIBD($v, v - k_1, \lambda_2$) for some λ_2. Note that the number of blocks and the replication number for the design is λ and as for a BOBIBD, at each location every element occurs $\frac{\lambda}{v}$, the replication number for BIBD(v, v-k_1, λ_2) is $(v - k_1)(\frac{\lambda}{v})$. Using the usual parametric relationships between design parameters, λ_2 is as given in the statement of the theorem. □

3. k= 3, k_1=2

For k=3 the only possible value of k_1 is 2. As λ=$\frac{\binom{k}{2\lambda_1}}{\binom{k-2}{k_1-2}}$, for a BOBIBD ($v$,3,$\lambda$,2,$\lambda_1$), λ=3λ_1 and hence λ has to be a multiple of 3.
There are two cases to consider: λ=6t+3 or λ=6t. It is well known that for block size $k = 3$ and λ=$6t + 3$, v has to be odd. On the other hand, BIBD($v, 3, 6t$) exists for any v. In other words, the necessary conditions for the existence of a BOBIBD(v,3,λ,2,λ_1) are:

λ	spectrum of BOBIBD(v,3,λ,2,λ_1)'s
λ≡1, 2(mod 3)	none
λ≡3(mod 6)	odd v
λ≡0(mod 6)	all v

Table 1

We might write a block {a,b,c} as abc and the context will indicate when the block is ordered.

Subcase λ=$6t + 3$

For this case, v must be odd, so v≡1, 3, 5(mod 6).
For v≡1,3(mod 6), a BIBD($v, 3, 1$) exists. Arrange 3 copies of each block {a,b,c} of the BIBD($v, 3, 1$) as {a,b,c}, {c,a,b}, and {b,c,a} to get a

BOBIBD$(v,3,3,2,1)$. A BOBIBD$(v,3,6t+3,2,2t+1)$ can be obtained by taking $(2t+1)$ copies of a BOBIBD$(v,3,3,2,1)$.

For $v\equiv5 \pmod 6$. Recall one can construct a PBD on v=$6t$+5 points with exactly one block, say {1,2,3,4,5}, and all other blocks of size 3 [**6**]. Order 3 copies of each of the triples as in the above paragraph. Combining with triples 123,412,251,314,531,154,235,342,425,543 of a BOBIBD$(5,3,2,2,1)$ we get a BOBIBD$(v,3,3,2,1)$. Here again, $(2t+1)$ copies yield the required BOBIBD$(v,3,6t+3,2,2t+1)$.

<u>Subcase $\lambda=6t$</u>

There is no restriction on v for $\lambda=6t$.

Even though one can use similar arguments again for v odd, a general construction gives the required designs for $\lambda=6t$ automatically.

Recall that one can construct a BIBD$(v,3,6)$ by an idempotent Quasigroup $(Q,\circ)$ of order v which exists for all order $v\geq 3$ where the collection of triples of the BIBD$(v,3,6)$ is {{a,b,a∘b} where a≠b∈Q}. Keeping the ordering of the elements in triples as it is, the properties of the Latin square guarantee that each pair {a,b} occurs at the location i,j, $1\leq i<j\leq 3$ in the triples exactly twice as required. Taking t copies of the design gives BOBIBD$(v,3,6t,2,2t)$. Hence we have:

THEOREM 9. *Necessary conditions given in Table* 1 *for the existence of* BOBIBD *with* $k=3$ *and* $k_1=2$ *are sufficient.*

4. $k=4$, $k_1=2$

From Theorem 3 and Theorem 4, we have the following corollary.

COROLLARY 3. *For* $k=4$, $k_1=2$,

(1) $\lambda_1=\frac{\lambda}{6}$, *therefore* $\lambda=6t$ *for some positive integer* t.
(2) *If* $\lambda=6(2n+1)$ *for some nonnegative integer* n, *then* v *is odd.* *If* $\lambda=6(2n)$ *for some nonnegative integer* n, *then there is no condition on* v.

Hence, the necessary condition for the existence of BOBIBD$(v,4,\lambda,2,\lambda_1)$ are:

λ	spectrum of BOBIBD$(v,4,\lambda,2,\lambda_1)$'s
$\lambda\equiv6 \pmod{12}$	all odd $v\geq5$
$\lambda\equiv0 \pmod{12}$	no condition on v

4.1. BOBIBD$(v,4,6,2,1)$ for odd $v\geq5$.

EXAMPLE 5. *One can construct a* BOBIBD$(5,4,6,2,1)$ *by deleting the first entries of all the blocks of Example* 1, BOBIBD$(5,5,10,2,1)$.

Using the above example and as BIBD$(v, 5, 1)$ exists for all v≡1,5(mod 20), we have

THEOREM 10. *A* BOBIBD$(v, 4, 6, 2, 1)$ *exists for all* v≡1,5(mod 20).

EXAMPLE 6. *A* BOBIBD$(7, 4, 6, 2, 1)$ *can be constructed with ordered difference sets* {7,1,2,4}, {7,2,4,1}, {7,4,1,2}.

EXAMPLE 7. *A* BOBIBD$(9, 4, 6, 2, 1)$ *is constructed below:*

7	1	2	3
8	2	3	1
9	3	1	2
1	4	5	6
2	5	6	4
3	6	4	5
4	7	8	9
5	8	9	7
6	9	7	8

6	1	4	7
9	4	7	1
3	7	1	4
4	2	5	8
7	5	8	2
1	8	2	5
5	3	6	9
8	6	9	3
2	9	3	6

3	1	5	9
4	5	9	1
8	9	1	5
1	2	6	7
5	6	7	2
9	7	2	6
2	3	4	8
6	4	8	3
7	8	3	4

9	1	6	8
2	6	8	1
4	8	1	6
7	2	4	9
3	4	9	2
5	9	2	4
8	3	5	7
1	5	7	3
6	7	3	5

THEOREM 11. *The necessary conditions* ($v \geq 5$ *and* v *odd*) *are sufficient for the existence of a* BOBIBD$(v, 4, 6t, 2, t)$ *except possibly for* 15*,* 27*,* 33*,* 39*,* 51*,* 75*,* 87*,* 95*,* 99*,* 111*, and* 115*.*

PROOF. A BOBIBD$(v, 4, 6, 2, 1)$ exists for {5,7,9} and hence for v≡1(mod 2) except possibly for (11-19), 23, (27-33), 39, 43, 51, 59, 71, 75, 83, 87, 95, 99, 107, 111, 113, 115, 119, 139, 179 [**1**]. Excluding the eleven exceptions listed in the theorem, one can construct BOBIBDs using Theorem 12 given below. Take t copies of BOBIBD$(v, 4, 6, 2, 1)$ to construct BOBIBD$(v, 4, 6t, 2, t)$ □

THEOREM 12. *For any prime p, ordered difference sets* $\{0,i,p\text{-}i,2i\}$*,* $i = 1, 2, \cdots, \frac{p-1}{2}$ *give* BOBIBD$(v, 4, 6, 2, 1)$.

PROOF. Differences from the ordered difference set $\{0,i,p\text{-}i,2i\}$ are i, $2i$, $3i$, i, i, $2i$ and as i runs through 1 to $\frac{p-1}{2}$ every difference from 1 to $\frac{p-1}{2}$ occurs exactly once for each pair of locations. □

4.2. BOBIBD(v,4,12,2,2) for all $v \geq 4$..

EXAMPLE 8. BOBIBD(4,4,12,2,2)

1	2	3	4
1	4	2	3
1	3	4	2
2	1	3	4
2	4	1	3
2	3	4	1

3	1	2	4
3	4	1	2
3	2	4	1
4	1	2	3
4	3	1	2
4	2	3	1

Note for $v \equiv 1,4 (\text{mod } 12)$, a BOBIBD can be constructed by rearranging the blocks of a BIBD$(v, 4, 1)$ according to the above example. Hence we have:

THEOREM 13. BOBIBD$(v, 4, 12, 2, 2)$ *exist for all* $v \equiv 1,4 (mod\ 12)$.

We need a BOBIBD$(6, 4, 12, 2, 2)$ which is given below:

1	2	6	4
1	3	2	6
1	4	5	2
1	5	3	4
1	6	4	2
2	1	4	5
2	3	6	5
2	4	3	6
2	5	1	3
2	6	5	3

3	1	2	4
3	2	5	1
3	4	6	1
3	5	4	6
3	6	1	4
4	1	5	6
4	2	3	5
4	3	1	2
4	5	6	2
4	6	2	5

5	1	6	3
5	2	1	6
5	3	4	1
5	4	2	3
5	6	3	1
6	1	3	2
6	2	4	3
6	3	5	4
6	4	1	5
6	5	2	1

To construct BOBIBD$(v, 4, 12, 2, 2)$ for all values of $v \geq 4$, we can extend the construction for BIBD$(v, 3, 6)$ by an idempotent Quasigroup of order v which exist for all required values of v's.

THEOREM 14. *Let* $L_1 = (Q, \circ_1)$*,* $L_2 = (Q, \circ_2)$ *be two mutually orthogonal idempotent Latin squares of order* v*. Then the set of blocks* $T = \{\{a, b, a \circ_1 b, a \circ_2 b\} : a \neq b \;\; a, b \in Q\}$ *gives a* BOBIBD$(v, 4, 12, 2, 2)$.

PROOF. Let $L_1 = (Q, \circ_1)$, $L_2 = (Q, \circ_2)$ be two mutually orthogonal idempotent Latin squares of order v which exist for all values of v except 2,3, and 6, (see Theorem 1).

Note that this construction generates $2\binom{n}{2} = n(n\text{-}1)$ blocks of size four which is the required number of blocks for a BIBD$(v, 4, 12)$.

For any $a \neq b$, we know that pair $\{a,b\}$ and pair $\{b,a\}$ occurs in the first two locations of the blocks at least twice.

Now consider the occurrences of the pair $\{a,b\}$ at the first and third location or second and third location. The third location entry is $a \circ_1 b$. It is clear that for some $x, y \in Q$, $a \circ_1 x = b$ and $y \circ_1 a = b$. Similarly, for some $w, z \in Q$, $b \circ_1 w = a$ and $z \circ_1 b = a$. Therefore the count of occurrences of the pair $\{a,b\}$ until now is at least 2+4=6.

Next we consider the occurrences of the pair $\{a,b\}$ at first and fourth or second and fourth locations. The fourth location entry is $a \circ_2 b$. Same argument can be used again in this case. Hence $\{a,b\}$ occurs at least 6+4=10 times.

Now since L_1 and L_2 are idempotent MOLS, there exists $p, q \in Q$, such that $p \circ_1 q = a$ and $p \circ_2 q = b$, and for some $r, s \in Q$ such that $r \circ_1 s = b$, $r \circ_2 s = a$. Hence $\{a, b\}$ occurs at least 10+2=12 times.

As the number of blocks is exactly the number of blocks needed for the design, $\lambda=12$.

The above counting for the index λ also shows that the construction produces BOBIBD$(v, 4, 12, 2, 2)$. □

Theorem 14 gives the construction of BOBIBD$(v, 4, 12t, 2, 2t)$ except for v=2,3, and 6. However, BOBIBD(6,4,12,2,2) is given above and as k=4 is bigger than 2 and 3, we have

THEOREM 15. *Necssary condition that* $v \geq 4$ *is sufficient for the existence of a* BOBIBD$(v, 4, 12\text{t}, 2, 2\text{t})$.

5. Block size k=4, k_1=3

THEOREM 16. *Necessary conditions for the existence of* BOBIBD$(v, 4, \lambda, 3, \lambda_1)$ *are* $\lambda = 2\lambda_1$ *(hence* λ *is even), and*

λ	λ spectrum
$\lambda \equiv 0 \pmod{12}$	all v
$\lambda \equiv 2,10 \pmod{12}$	$v \equiv 1 \pmod 6$
$\lambda \equiv 6 \pmod{12}$	all odd v
$\lambda \equiv 4,8 \pmod{12}$	$v \equiv 1 \pmod 3$

PROOF. Necesscary conditions for BIBD$(v, 4, \lambda)$ imply $\lambda(v-1) = 3r$ and $\frac{\lambda(v)(v-1)}{12} = b$. □

5.1. $\lambda \equiv$ 0,6(mod 12). We have proved for BOBIBD$(v, 4, 6, 12t+6, 2, 2t+1)$ exists for all odd v and BOBIBD$(v, 4, 12t, 2, 2t)$ exists for any $v \geq 4$ and hence we have the following result.

THEOREM 17. *Necessary conditions are sufficient for* BOBIBD$(v, 4, 12t+6, 3, 6t+3)$.

THEOREM 18. *Necessary conditions are sufficient for* BOBIBD$(v, 4, 12t, 3, 6t)$.

EXAMPLE 9. *Using the Self-Orthogonal Latin squares of order* 7 *given in* [**4**] *, we can construct the following* BOBIBD(7,4,6,3,3)*:*

1	2	7	6
1	3	6	4
1	4	5	2
2	3	1	7
2	4	7	5
2	5	6	3
3	4	2	1
3	5	1	6
3	6	7	4
4	5	3	2
4	6	2	7

4	7	1	5
5	6	4	3
5	7	3	1
5	1	2	6
6	7	5	4
6	1	4	2
6	2	3	7
7	1	6	5
7	2	5	3
7	3	4	1

5.2. $\lambda \equiv 2,10 \pmod{12}$.

THEOREM 19. BOBIBD(7,4,2,3,1) *does not exist.*

PROOF. Assume the design exists. Without loss of generality, let the first column and first row be:

2	3	4	1
			2
			3
			4
			5
			6
			7

.

Since 1 is already paired with 2, 3, 4 once and due to the facts that one already appears in the fourth column, without loss of generality, assume one is distributed diagonal from row 5 to row 7 as shown below,

2	3	4	1
			2
			3
			4
1			5
	1		6
		1	7

.

Note that the element 2 cannot be placed in row 5 and column 1. In addition, 5 has to occur in row 6 or row 7, but 5 cannot be in the first column. Hence we have two cases to consider because we can place 2 into row 6 and place 5 into row 7 or place 5 into row 6 and place 2 into row 7, for both cases, 2 or 5 has to be placed directly next to 1 in the 3rd or the 2nd column

2	3	4	1
			2
			3
			4
1			5
	1	2/5	6
	5/2	1	7

.

Consider Case 1: where 2 and 5 are placed in row 6 and row 7 respectively.

2	3	4	1
			2
			3
			4
1			5
	1	2	6
	5	1	7

As we can see from above, 7 is forced to be placed into row 6, 4 and 6 are placed into row 5 and finally 3 is placed into row 7

2	3	4	1
			2
			3
			4
1	4	6	5
7	1	2	6
3	5	1	7

.

Back to row 2, 7 and 3 are placed as follow

2	3	4	1
	7	3	2
			3
			4
1	4	6	5
7	1	2	6
3	5	1	7

.

In row 3, 6, 2, and 5 are forced to be placed as shown below,

2	3	4	1
	7	3	2
			3
6	2	5	4
1	4	6	5
7	1	2	6
3	5	1	7

.

In the final configuration shown below, zero denotes the locations where the conflict occurs,

2	3	4	1
5	7	3	2
4	0	0	3
6	2	5	4
1	4	6	5
7	1	2	6
3	5	1	7

.

Case 1

The final configuration with conflict for case 2 is shown below as in Case 1,

2	3	4	1
5	4	6	2
7	5	2	3
0	0	0	4
1	7	3	5
4	1	5	6
6	2	1	7

Case 2

.

□

THEOREM 20. *The blocks of a* BIBD(v,4,2) *can not be ordered to contruct a* BOBIBD(v,4,2,3,1) *if there exists two identical or two blocks with* 3 *common points.*

PROOF. Suppose the intersection number of two blocks is 4, i.e. two blocks are identical. Let b_1={a,b,c,d} $= b_2$ be two blocks of the BIBD(v,4,2). Without loss of generality, we only rearrange b_2, and hence we have the following four cases to consider:

(1) Consider configuration below:

a	b	c	d
a			

where we placed a in the first location, no matter which way we rearrange b,c,d, for some locations (i_1,i_2,i_3) a pair appears more than once.

(2) Consider b in the first location of second block, the same argument we can use as in Case 1.

(3) Consider c in the first location as displayed below:

a	b	c	d
c	d		

The only possible entry at the second location is d, and no matter how we place a and b, there exist three locations of b_1 and b_2 where a pair appears twice.

(4) Consider d in the first location and c in the second location. The same argument can be made as in Case 3.

Similarly one can show that if two blocks have 3 common element then it is impossible to order the blocks to get a BOBIBD(v,4,2,3,1). □

5.3. $\lambda\equiv$4,8(mod 12). Construction for k=3, λ=2, v=3n+1

The 3n+1 Construction. See for example [**6**] Let (Q, $\circ$) be an idempotent (not necessarily commutative) quasigroup of order n and set S=$\{\infty\}\cup(Q\times\{1,2,3\})$. Define a collection of triples T as follows:

Type 1: The four triples $\{\infty,(x,1),(x,2)\}$, $\{\infty,(x,2),(x,3)\}$, $\{\infty,(x,1),(x,3)\}$, $\{(x,1),(x,2),(x,3)\}$ belong to T for every $x\in Q$ (note: these 4 triples form a 2-fold triple system of order 4) and

Type 2: If $x\neq y$, the six triples $\{(x,1),(y,1),(x\circ y,2)\}$, $\{(y,1),(x,1),(y\circ x,2)\}$, $\{(x,2),(y,2),(x\circ y,3)\}$, $\{(y,2),(x,2),(y\circ x,3)\}$, $\{(x,3),(y,3),(x\circ y,1)\}$, $\{(y,3),(x,3),(y\circ x,2)\}$ belong to T.

Then (S,T) is a 2-fold triple system of order $3n+1$.

The above construction can be generalized easily to obtain a BOBIBD(v,4,4,3,2).

Theorem 21. *If two idempotent* MOLS *of order n exist, then* BOBIBD(v=3n+1 ,4,4,3,2) *exists. Therefore necessary condition for* $\lambda\equiv$4(mod 12) *hold except possibly for* v=7,10,19.

PROOF. Let L_1=(X, $\circ_1$) and L_2=(X, $\circ_2$) be two idempotent MOLS, and set S={∞}∪(X×{1,2,3}). Define a collection of quadruples T as follows:

Type 1: Four copies of the quadruple {∞,(x,1),(x,2),(x,3)} belong to T for every $x\in X$,and

Type 2: If $x\neq y$, the quadruples {(x,1),(y,1),($x\circ_1 y$,2),($x\circ_2 y$,2)}, {(y,1),(x,1),($y\circ_1 x$,2),($y\circ_2 x$,2)}, {(x,2),(y,2),($x\circ_1 y$,3),($x\circ_2 y$,3)}, {(y,2),(x,2),($y\circ_1 x$,3),($y\circ_2 x$,3)}, {(x,3),(y,3),($x\circ_1 y$,1),($x\circ_2 y$,1)}, {(y,3),(x,3),($y\circ_1 x$,1),($y\circ_2 x$,1)} belong to T.

It is easy to see that k=4, since all the blocks are quadruple. Moreover, there are $4n$ blocks of type 1 and $6\binom{n}{2}$ blocks of type 2. Therefore we have a total of $(3n+1)n$ blocks, equal to the total numbers of blocks required for a BIBD$(3n+1,4,4)$. Next we want to show that each pair occurs at least 4 times. Let (x,i), (y,j) be any pairs. There are three cases to consider.

- Suppose that x=y, $i\neq j$. Then four copies of Type 1 quadruples {∞,(x,1),(x,2),(x,3)} contain (x,i) and (x,j) four times.

- Suppose that i=j. Then $x\neq y$ whence, {(x,i),(y,i),($x\circ_1 y$,(i+1)(mod 3),($x\circ_2 y$,(i+1)(mod 3)},{(y,i),(x,i),($y\circ_2 x$,(i+1)(mod 3)},{($y\circ_2 x$,(i+1)(mod 3)} $\in T$ are two blocks containing (x,i) and (y,i). Now we will use orthogonality of latin squares, there are r,s,u,w in X such that $r\circ_1 s$=x, $r\circ_2 s$=y, $u\circ_1 w$=y, and $u\circ_2 w$=x. Therefore {(r,(i-1)(mod 3), (s,(i-1)(mod 3)), (x,i), (y,i)}, {(u,(i-1)(mod 3), (w,(i-1)(mod 3)), (y,i), (x,i)} are the other two blocks containing (x,i) and (y,i).

- Finally suppose that $x\neq y$ and $i\neq j$. Without loss of generality, assume that i=1 and j=2. since (L_1, $\circ_1$) and (L_2, $\circ_2$) are Latin square, $x\circ_1 a$=y and $x\circ_2 b$=y for some a,$b\in X$. Since L_1 and L_2 are idempotent MOLS and $x\neq y$, it must be that $a\neq x$ and $b\neq x$. Therefore {(x,1),(a,1),($x\circ_1 a$=y,2),($x\circ_2 a$,2)} and {(x,1),(b,1),($x\circ_1 b$,2),($x\circ_2 b$=y,2)} are Type 2 quadruple in T and contains (x,i) and (y,j).

Each Type 1 quadruple appears four times, hence we can rearrange those same blocks as those in BOBIBD(4,4,4,3,2). For Type 2 quadruple, we have four cases to consider. Suppose {(x,i),(y,i),($x\circ_1 y$,j),($x\circ_2 y$,j)}, where j=(i+1)(mod 3). From the (3n+1) construction for triple system the set of subblocks {(x,i),(y,i),($x\circ_1 y$,j)} is a BIBD$(3n+1,3,2)$. Similarly if we select 1^{st},2^{nd},and 4^{th} location elements of each ordered block, BIBD$(3n+1,3,2)$ will follow. Now suppose we select 1^{st},3^{rd},and 4^{th} then we have $\{(x,i),(x\circ_1 y,j),(x\circ_2 y,j)\}$, by the properties of Idempotent MOLS, we get a BIBD$(3n+1,3,2)$. Same result will follows if we select 2^{nd},3^{rd},and 4^{th}. □

THEOREM 22. *Necessary Conditions are sufficient for BOBIBD*$(3n+1,4,8,3,4)$.

PROOF. BIBD$(3n+1,4,2)$ exists and BOBIBD$(4,4,4,3,2)$ exists. So take four copies of each block and arrange as BOBIBD$(4,4,4,3,2)$. □

COROLLARY 4. *Necessary conditions are sufficient for BOBIBD*$(3n+1,4,12t+8,3,6t+4)$.

ACKNOWLEDGEMENT 1. *Authors wish to express their sincere gratitude for the support received from the College of Charleston SURF grant and the College of Charleston Summer R and D grant. The authors would like to thank an anonymous mathematician for the comments which improved the presentation and terminology used in the paper including the suggestion and wording for the alternate definition.*

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