A non-existence result and large sets for Sarvate-Beam designs

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Abstract. It is shown that for $2 \leq t \leq n - 3$, a strict $t$-SB$(n, n - 1)$ design does not exist, but for $n \geq 3$, a non-strict 2-SB$(n, n - 1)$ design exists. The concept of large sets for Steiner triple systems is extended to SB designs and examples of a large sets for SB designs are given.

1. Introduction

Stanton [9] renamed a type of block design that was introduced in [7] as Sarvate-Beam Triple Systems (SB Triple Systems). In addition, Stanton obtained several interesting results and raised questions on enumeration and existence, see [10], [11], [12] and [13]. Some of these questions are solved by Hein and Li [5] as well as Bradford, Hein and Pace [1]. In general, an SB design is a block design in which every pair occurs in a different number of blocks. Below is a formal definition:

Definition 1. A Sarvate-Beam design, $SB(v,k)$, consists of a $v$-set $V$ and a collection of $k$-subsets, called blocks, of $V$ such that each distinct pair of elements in $V$ occurs with different frequencies i.e., in a different number of blocks. A strict $SB(v,k)$ design is a design where for every $i$, $1 \leq i \leq (v\choose{2})$, exactly one pair occurs exactly $i$ times.

Example 1. A strict $SB(4,3)$ on $\{1,2,3,4\}$ consists of the following blocks:

$\{1,2,4\}, \{1,3,4\}, \{1,3,4\}, \{2,3,4\}, \{2,3,4\}, \{2,3,4\}.$

Although the general existence question of strict SB block designs is still an open question, it has been proven that the necessary conditions are
sufficient for $k = 3$ by Dukes [3] except for some finite number of exceptions. On the other hand, Ma, Chang and Feng [6] have proved that the necessary conditions are sufficient for $k = 3$. Moreover, SB matrices have been studied by Dukes, Hurd and Sarvate [4]. The following definition and result appear in [8]:

**Definition 2.** A $t$-SB$(v,k)$ design is a collection, $B$, of $k$-subsets of a $v$-set such that each $t$-subset of $V$ occurs a distinct number of times. In a strict $t$-SB design, for each $i$, $1 \leq i \leq \binom{v}{t}$, there is exactly one $t$-subset which occurs in $i$ blocks.

**Theorem 1.** A strict $t$-SB$(v,k)$ exists only if
\[
\binom{v}{t} \mid \binom{v}{t}(\frac{v}{t}+1).
\]

2. **Non-existence result**

The following result is known [8]:

**Theorem 2.** For $n > 4$, a strict $(n - 2)$-SB$(n,n - 1)$ does not exist.

We prove the following result:

**Theorem 3.** For $n > 4$, a strict $t$-SB$(n,n - 1)$ does not exist for $2 \leq t \leq n - 3$.

**Proof.** Let us denote the frequency of an $s$-subset, $\{a_1, a_2, ..., a_s\}$, in the design by $f(a_1, ..., a_s)$. Let $B_i = \{1, 2, ..., n\} - \{i\}$, $i = 1, 2, \cdots, n$, be the $n$ subsets of size $n - 1$ of $\{1, 2, \cdots, n\}$. Let $F(B_i)$ denotes the frequency of the block $B_i$ in the design if it exists. Without loss of generality, assume that the $t$-subset $\{1, 2, ..., t\}$ appears exactly once and let $B_n = \{1, 2, ..., t, ..., n - 1\}$ be the block containing $\{1, 2, \cdots, t\}$ that appears exactly once. Observe that there are $n - t$ sets, $B_{t+1}, B_{t+2}, ..., B_n$, among $B_1, B_2, ..., B_{n-1}, B_n$ containing $\{1, 2, ..., t\}$, and $n - t + 1$ sets, $B_1, B_{t+1}, ..., B_n$, containing $\{1, 2, ..., t - 1\}$. As the frequency of $\{1, 2, ..., t\}$ is one and $F(B_n) = 1$, it follows that $F(B_{t+1}) = F(B_{t+2}) = ... = F(B_{n-1}) = 0$. Hence, there exists only one other set, $B_t$, which contains $\{1, 2, ..., t - 1\}$ but not $\{1, 2, ..., t\}$ whose frequency (say $\phi$) may be greater than one in the design. This is the only set other than $B_n$ which contains $\{1, 2, ..., t - 1, x\}$ and $\{1, 2, ..., t - 1, y\}$, where $x, y \in \{t, ..., n\}$ and $x \neq y$. Hence $f(1, 2, ..., t - 1, x) = \phi + 1 = f(1, 2, ..., t - 1, y)$, which is a contradiction. □

The following example is illustrative:

**Example 2.** A strict $3$-SB$(6,5)$ does not exist. First note that the design parameters satisfy Theorem 1. There are 6 subsets $\{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 6\}, \{1, 2, 3, 5, 6\}, \{1, 2, 4, 5, 6\}, \{1, 3, 4, 5, 6\}, \{2, 3, 4, 5, 6\}$. Without loss of generality, assume the $3$-subset $\{1, 2, 3\}$ occurs exactly once in
the block \{1, 2, 3, 4, 5\}. Note that we cannot have blocks \{1, 2, 3, 4, 6\} and \{1, 2, 3, 5, 6\} in this design since we want \{1, 2, 3\} to appear exactly once. Therefore the remaining blocks must be some multiple copies of the sets \{1, 2, 4, 5, 6\}, \{1, 3, 4, 5, 6\}, and \{2, 3, 4, 5, 6\}.

Let \(a, b,\) and \(c\) denote the frequency of the blocks \{1, 2, 4, 5, 6\}, \{1, 3, 4, 5, 6\}, and \{2, 3, 4, 5, 6\} respectively, if the design exists. Note \(f(1, 2, 4) = 1 + a = f(1, 2, 5)\), which is a contradiction.

3. Non-strict 2-SB\((n, n−1)\) designs

Although strict 2-SB\((n, n−1)\) designs do not exist for any \(n\), non-strict 2-SB\((n, n−1)\) designs exist for all \(n \geq 3\):

**Lemma 1.** A non-strict \(t\)-SB\((n, n−1)\) design is also a non-strict \((t−1)\)-SB\((n, n−1)\) design if \(n−1 \geq 2t−2\).

**Proof.** Suppose the block \(B_i = \{1, 2, \cdots, n\} - \{i\}\) occurs \(f_i\) times in the non-strict \(t\)-SB\((n, n−1)\) design. A \((t−1)\)-set \(\{i_1, i_2, \cdots, i_{t−1}\}\) occurs in \(b(\sum_{i=1}^{t−1} f_i)\) blocks, where \(b\) is the total number of blocks of the non-strict \(t\)-SB\((n, n−1)\) design. If the design is not a non-strict \((t−1)\)-SB\((n, n−1)\) design, then there exists at least two distinct \((t−1)\)-sets, \(\{a_1, a_2, \cdots, a_{t−1}\}\) and \(\{b_1, b_2, \cdots, b_{t−1}\}\) both occurring the same number of times (say \(\mu\)) in the design. As \(2t−2 = n−1\), there exists an element \(a\) in \(\{1, 2, \cdots, n\}\) but not in the union of \(\{a_1, a_2, \cdots, a_{t−1}\}\) and \(\{b_1, b_2, \cdots, b_{t−1}\}\). Consider the \(t\)-sets \(\{a, a_1, a_2, \cdots, a_{t−1}\}\) and \(\{a, b_1, b_2, \cdots, b_{t−1}\}\). Clearly both occur in \(\mu − f_a\) blocks of the non-strict \(t\)-SB\((n, n−1)\) design which is a contradiction. \(\square\)

In general a \(t\)-SB\((n, k)\) design need not be a \((t−1)\)-SB\((n, k)\) design as shown below:

**Example 3.** Let \(V = \{1, 2, 3, 4\}\). The collection of blocks with \(t\) copies of \(\{1, 2\}\), one copy of \(\{1, 3\}\), four copies of \(\{1, 4\}\), two copies of \(\{2, 3\}\), three copies of \(\{2, 4\}\) and \(s\) copies of \(\{3, 4\}\) for any distinct values of \(s\) and \(t\) different from 1, 2, 3, and 4 provides a 2-SB\((4, 2)\) design. The design is a strict 2-SB\((4, 2)\) when \(\{s, t\} = \{5, 6\}\). Note that the elements 1 and 2 both have the replication number \(t + 5\) and hence the design is not a 1-SB\((4, 2)\) design.

**Theorem 4.** A non-strict 2-SB\((n, n−1)\) design exists for every positive integer \(n \geq 3\).

**Proof.** Let the set of elements be \(\{1, 2, \cdots, n\}\) for a design on \(n\) elements. The proof is based on induction. For \(n = 3\), a non-strict 2-SB\((3, 2)\) design can be easily constructed. Suppose we have a non-strict 2-SB\((n, n−1)\) for some value of \(n\) with \(b\) blocks. We construct a 2-SB\((n+1, n)\) design containing \(2b\) blocks using the blocks of the non-strict 2-SB\((n, n−1)\)
design and the set \( \{1, 2, \cdots, n\} \) as follows. First we construct \( b \) blocks by adding the element \( n+1 \) into each block of the non-strict 2-SB\((n, n-1)\) design. Note that in these blocks each element from 1 to \( n \) occurs different number of times, therefore the pairs \( \{n+1, i\} \) occur different number of times. We complete the construction of non-strict 2-SB\((n+1, n)\) design by including \( b \) copies of the set \( \{1, 2, \cdots, n\} \). The maximum number of times a pair \( \{n+1, i\} \) may have occurred is \( b \), and minimum number of times a pair \( \{i, j\} \), \( 1 \leq i < j \leq n \), occurs in the non-strict 2-SB\((n, n-1)\) design is one. Hence, all pairs occur a different number of times in these \( 2b \) blocks of the non-strict 2-SB\((n+1, n)\) design. \( \square \)

4. Large sets

**Definition 3.** A triple system \((V, B)\) is a set \( V \) of \( v \) elements together with a collection \( B \) of 3-subsets (called blocks or triples) of \( V \) with the property that every 2-subset of \( V \) occurs in exactly \( \lambda \) blocks. The size of \( V \) is the order of the triple system. It is also denoted by \( TS(v, \lambda) \), or Steiner triple system, \( STS(v) \), when \( \lambda = 1 \).

**Definition 4.** Let \((V, B)\) and \((V, D)\) be two \( STS(v) \)’s. Their intersection size is \( |B \cap D| \). They are disjoint when their intersection size is zero. A set of \( (v-2) \) \( STS(v) \)’s, \( \{(V, B_i) : i=1, \ldots, v-2\} \), is a large set if any two distinct systems from the set are disjoint.

In other words, the set of all 3-subsets of a \( v \)-set is partitioned into \( v-2 \) \( STS(v) \)’s. It is known that large sets for triple systems exist for all \( v \equiv 1,3 \pmod{6} \) except for \( v = 7 \) [2].

The analogous question to the large set for triple system with respect to SB triple systems can be formulated using the following definition:

**Definition 5.** Let \( V \) be a \( v \)-set. A family of SB\((v, k)\) designs on \( V \), say \( B=\{B_1, B_2, \ldots, B_n\} \), is a large set with multiplicity \( s \) if \( \bigcup_{i=1}^{n} B_i \) gives \( s \) copies of the set of all \( k \)-subsets of \( V \) for some integer \( s \) and if there is another family of SB\((v, k)\) designs \( C=\{C_1, C_2, \ldots, C_m\} \) where \( \bigcup_{i=1}^{m} C_i \) contains \( t \) copies of all \( k \)-subsets of \( V \), then \( s \leq t \).

Simple counting gives the following result:

**Theorem 5.** Suppose the multiplicity for the large set for a \( SB(v, k) \) is \( s \) and let the size of the large set be \( n \). Then \( s(v) = \frac{\binom{v}{k} \left( \frac{\binom{v}{k}+1}{2} \right)}{s_k} \times n \); hence a necessary condition for the existence of a large set for strict \( SB(v, k) \) is \( \frac{\binom{v}{k} \left( \frac{\binom{v}{k}+1}{2} \right)}{s_k} \). \( \frac{\binom{v}{k} \left( \frac{\binom{v}{k}+1}{2} \right)}{s_k} \)

**Corollary 1.** For \( k = 3 \), \( \frac{\binom{v}{3} \left( \frac{\binom{v}{3}+1}{6} \right)}{s_3} \).
The following example will clarify the definition:

**Example 4.** Consider the set \( V = \{1, 2, 3, 4\} \). We have the following 4 strict SB\((4, 3)\)'s.

- \( B_1 = \{\{1,2,4\}, \{1,3,4\}, \{1,3,4\}, \{2,3,4\}, \{2,3,4\}, \{2,3,4\}\} \)
- \( B_2 = \{\{1,2,3\}, \{1,2,3\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}, \{2,3,4\}\} \)
- \( B_3 = \{\{1,2,3\}, \{1,2,3\}, \{1,2,4\}, \{1,2,4\}, \{1,2,4\}, \{2,3,4\}\} \)
- \( B_4 = \{\{1,2,3\}, \{1,2,4\}, \{1,2,4\}, \{1,3,4\}, \{1,3,4\}, \{1,3,4\}\} \).

When we take the multi-union \( B_1 \cup B_2 \cup B_3 \cup B_4 \), we get a multi-set where each of the blocks \( \{1, 2, 3\} \), \( \{1, 2, 4\} \), \( \{1, 3, 4\} \), and \( \{2, 3, 4\} \) occurs 7 times. Indeed 7 is the multiplicity for SB\((4, 3)\), because there are only 4 distinct blocks and SB\((4, 3)\) has 7 blocks, if the multiplicity is \( s \), then \( 4 \times s = 7 \times n \) for some integer \( n \). Therefore \( \{B_1, B_2, B_3, B_4\} \) is the large set for SB\((4, 3)\), and as the SB\((4, 3)\) is unique, the large set is unique up to isomorphism.

**Example 5.** A set of SB\((6, 3)\) designs such that the multi-union of the collections of blocks has multiplicity \( t = 10 \) is given below, however this may not be a large set. The reason is that we obtained these designs by taking isomorphic copies of a single SB\((6, 3)\) design, but according to [5], there are 48,843 non-isomorphic restricted SB\((6, 3)\), and a total of 16, 444, 250 (restricted and non-restricted) SB\((6, 3)\) designs. What we can claim is that using this particular design, the multiplicity cannot be less than 5.

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4.1. Large sets for $k = 2$. Let us consider the following two examples:

**Example 6.** A strict SB(3, 2) design with blocks \{\{1, 2\}, \{1, 3\}, \{1, 3\}, \{2, 3\}, \{2, 3\}, \{2, 3\}\}.

**Example 7.** Another strict SB(3, 2) design with blocks \{\{1, 2\}, \{1, 2\}, \{1, 2\}, \{1, 3\}, \{1, 3\}, \{2, 3\}\}.

The union of these designs is a multi-set that contains each 2-subset with a multiplicity of 4. In fact, these designs form a large set. This simple observation leads to the following result:

**Theorem 6.** Large sets with multiplicity \(\binom{v}{2} + 1\) containing exactly two SB\(v, 2\)'s exist for all \(v \geq 2\).

**Proof.** Let the 2-subsets of a v-set \(V\) be \(\{b_1, b_2, ..., b_{\binom{v}{2}}\}\). Without loss of generality, let the first SB\(v, 2\), \(B_1\), contain blocks \(b_i\) with frequency \(i\). Now construct a second SB\(v, 2\), \(B_2\), where \(b_i\) occurs with frequency \(\binom{\binom{v}{2}}{i} + 1 - i\). It follows that we have a partition \(\{B_1, B_2\}\) of the collection of the 2-subsets of \(V\) with multiplicity \(\binom{\binom{v}{2}}{i} + 1\). \(\square\)

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[5] D. W. Hein and P. C. Li, Sarvate-Beam triple systems for \(v = 5\) and \(v = 6\), to be submitted.


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