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On some partial line graphs of a hypergraph and the associated matroid

Philippe Jégou

Laboratoire d'Informatique, de l'Université de Provence, UFR-MIM, 3 place Victor Hugo, 13331 Marseille, France

Marie-Catherine Vilarem

LIRMM 860 Rte de Saint-Priest 34090 Montpellier, France

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Abstract

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In this paper, we define for a hypergraph $H=(X, \mathscr{E})$ a class of partial graphs of its line graph $\mathrm{GR}(H)$; these graphs are called intergraphs and verify the following property:

> for each intergraph $G=(\mathscr{E}, U)$ we have: $\forall E_i, E_j \in \mathscr{E} | E_i \cap E_j \neq \emptyset$, there exists in G a chain $(E_i=E_1, E_2, \ldots, E_q=E_j)$ such that $\forall k$, $1 \leqslant k < q$, $E_i \cap E_j \subseteq E_k \cap E_{k+1}$.

We show that all the intergraphs minimal w.r.t. inclusion have the same number of edges. Moreover, we show that they are the bases of a matroid. These properties allow us to define a cyclomatic number for a hypergraph, and we show some connections with a previous work on hypergraph cyclicity [Acharya and Las Vergnas (1982)].

In the last section we give an application of these results to constraint networks.

0. Introduction

In this paper, we study a class of partial graphs of the line graph of a hypergraph. We call these graphs intergraphs. They were first defined in the context of relational databases theory as 'qual graphs' and were used to define databases schemes having good properties [2]. Besides, they are also useful in the context of general constraint networks, as they provide an equivalent binary representation [11].

Correspondence to: Philippe Jégou, Laboratoire d'Informatique, de l'Université de Provence, UFR-MIM, 3 place Victor Hugo, 13331 Marseille, France.

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The main results of this paper deal with minimal intergraphs, that allow minimal representations. We show that all the minimal intergraphs of a hypergraph have the same number of edges, and that they are the bases of a matroid.

We then give anew definition of the cyclomatic number of a hypergraph, followed by some remarks on the connections with previous definitions of a cyclomatic number for hypergraphs.

In the last section, we give briefly the applications of these results to the resolution of general constraint networks.

1. Intergraphs

Definition 1.1 (Berge [3]). Let $H=(X, \mathscr{E})$ be a hypergraph; the *labelled line graph of H* is the graph $\mathrm{GR}(H)=(\mathscr{E}, F, v_F)$, where v_F is a labelling of the edges such that (see Fig. 1):

- $F=\{\{E_i, E_j\} \subset \mathscr{E} \mid i \neq j \text{ and } E_i \cap E_j \neq \emptyset\}$,
- $v_F : F \to \mathscr{P}(X)$,
- $v_F(\{E_i, E_j\})=E_i \cap E_j$.

The labelled line graph allows the reconstruction of the original hypergraph, with the exception of vertices belonging to a unique edge of the hypergraph. In fact, we do not even need all the edges of the labelled line graph to reconstruct the original hypergraph: in the above example, $\{E_1, E_2\}$ is unnecessary, as the labelling of $\{E_1, E_3\}$ and $\{E_3, E_2\}$ allows us to see that $E_1 \cap E_2$ contains $\{X_2\}$.

Intuitively, the intergraphs of H are all the partial graphs of its labelled line graph from which we can reconstruct H.

Definition 1.2 (Bernstein and Goodman [5]). Let $H=(X, \mathscr{E})$ be a hypergraph; an *intergraph of H* is a graph $G(H)=(\mathscr{E}, U, v_U)$, where v_U is a labelling of the edges, such that:

- $G(H)=(\mathscr{E}, U, v_U)$ is a partial graph of $\mathrm{GR}(H)=(\mathscr{E}, F, v_F)$ (i.e. $U \subseteq F$),
- $v_U : U \to \mathscr{P}(X)$ is the restriction of v_F to U,
- $\forall E_i, E_j \in \mathscr{E}$, if $E_i \cap E_j \neq \emptyset$, there exists in $G(H)$ a chain $(E_i=E_1, E_2, \ldots, E_q=E_j)$ such that $\forall k, 1 \leqslant k < q$, $E_i \cap E_j \subseteq v_U(\{E_k, E_{k+1}\})$ (we denote this property as $\mathbb{P}$).

For a given hypergraph there can be several intergraphs (see Fig. 2); so, in the sequel, $\mathscr{C}(H)$ denotes the set of all intergraphs of the hypergraph H. The labelled line graph of H is the maximal element of $\mathscr{C}(H)$ under inclusion.

2. Minimal intergraphs: definition and first properties

Definition 2.1. Let $H=(X, \mathscr{E})$ be a hypergraph; $G_m(H)$ is a *minimal intergraph of H* if $G_m(H)=(\mathscr{E}, U_m, v_{U_m}) \in \mathscr{C}(H)$, and if U_m is minimal w.r.t inclusion (i.e. no partial graph of $G_m(H)$ belongs to $\mathscr{C}(H)$).

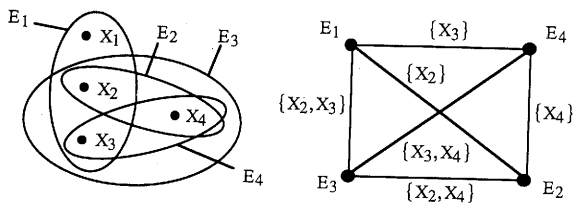


Fig. 1. A hypergraph and its labelled line graph.

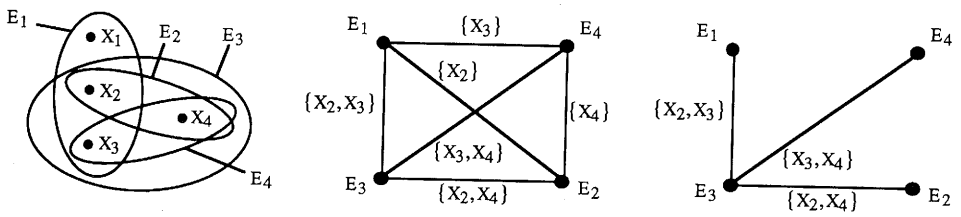


Fig. 2. A hypergraph and 2 of its 8 intergraphs.

Notation. $\mathscr{C}_m(H)$ denotes the class of minimal intergraphs (see Fig. 3) of the hypergraph H; by definition, $\forall H, \mathscr{C}_m(H) \subseteq \mathscr{C}(H)$.

The main result of this section is that all minimal intergraphs of a hypergraph have the same number of edges.

Notations. Given $H=(X, \mathscr{E})$ and $G(H)=(\mathscr{E}, U, v_U) \in \mathscr{C}(H)$, we introduce the following notation (see Figs. 4 and 5):

- $\mathscr{I}=\{E_i \cap E_j \mid E_i \cap E_j \neq \emptyset\}$.
- For $A \in \mathscr{I}$, $\mathbb{S}_A=\{E_i \in \mathscr{E} \mid A \subseteq E_i\}$.
- Let $A_1, A_2, \ldots, A_h$ be a total order on $\mathscr{I}$ compatible with $\supset$ (i.e. $\forall A_i, A_j \in \mathscr{I} \mid A_i \not\subseteq A_j \Rightarrow j<i$).
- $U^k=\{e \in U \mid v_U(e)=A_i, i \leqslant k\}$.

As easy consequences of these definitions, we have:

- $A_j \subseteq A_i \Leftrightarrow \mathbb{S}_{A_i} \subseteq \mathbb{S}_{A_j}$,
- $i \leqslant j \Rightarrow U^i \subseteq U^j$.

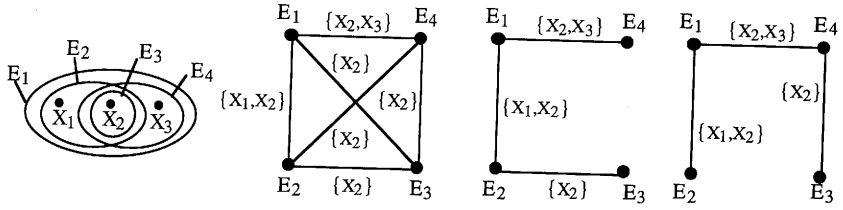
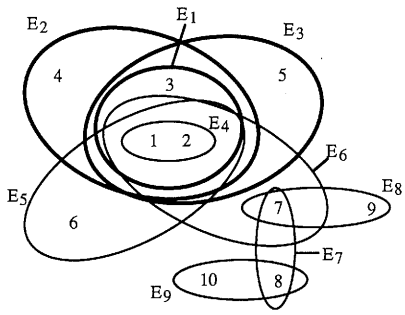


Fig. 3. A hypergraph and three of its intergraphs. Among them, two are minimal intergraphs.



$E_1=\{1,2,3\}$ $A_1=\{1,2,3\}$
$E_2=\{1,2,3,4\}$ $A_2=\{1,2\}$
$E_3=\{1,2,3,5\}$ $A_3=\{7\}$
$E_4=\{1,2\}$ $A_4=\{8\}$
$E_5=\{1,2,6\}$
$E_6=\{1,2,7\}$ $\mathcal{S}_{A_1}=\{E_1,E_2,E_3\}$
$E_7=\{7,8\}$ $\mathcal{S}_{A_2}=\{E_1,E_2,E_3,E_4,E_5,E_6\}$
$E_8=\{7,9\}$ $\mathcal{S}_{A_3}=\{E_7,E_8\}$
$E_9=\{8,10\}$ $\mathcal{S}_{A_4}=\{E_7,E_9\}$

Fig. 4. Example for the notations A_i and $\mathcal{S}_{A_j}$.

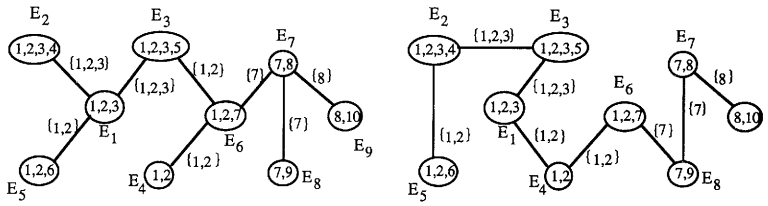


Fig. 5. Two minimal intergraphs of the hypergraph in Fig. 4. The labelling of each vertex E_i is the associated set E_i.

In what follows, we consider, without loss of generality, a connected hypergraph H, along with a total order on $\mathcal{I}$ compatible with $\supset$.

Lemma 2.2. *Let* $G=(\mathcal{E}, U, v_U)\in\mathcal{C}(H)$; *if* $E_{i_1}, E_{i_2}\in\mathcal{E}$ *are such that* $E_{i_1}\cap E_{i_2}=A_i$, *then there is a chain* $(E_0=E_{i_1},\ldots,E_k,\ldots,E_q=E_{i_2})$ *in* G *such that* $\forall k,\ 0\leqslant k<q$, $A_i\subseteq v_U(\{E_k, E_{k+1}\})$; *furthermore*, $\forall k,\ 0\leqslant k<q$, $\{E_k, E_{k+1}\}\in U^i$.

Proof. $A_i\subseteq v_U(\{E_k, E_{k+1}\})=A_r$ and, by definition of the order, $r\leqslant i$; therefore, we have $\{E_k, E_{k+1}\}\in U^i$. □

In what follows, $G^i(\mathcal{S}_{A_j})$ denotes the subgraph of $(\mathcal{E}, U^i)$ induced by $\mathcal{S}_{A_j}$.

Lemma 2.3. $G=(\mathcal{E}, U, v_U)\in\mathcal{C}(H)\Leftrightarrow\forall i,\ 1\leqslant i\leqslant h,\ G^i(\mathcal{S}_{A_i})$ *is connected* (*see Fig.* 6).

Proof. (⇒): $\forall E_{i_1}, E_{i_2}\in\mathcal{S}_{A_i}$, $E_{i_1}\cap E_{i_2}=A_j\supseteq A_i$, and there is a chain $(E_0=E_{i_1},\ldots,E_k,\ldots,E_q=E_{i_2})$ *in* G such that:

- $\forall k,\ 0\leqslant k\leqslant q$, $A_j\subseteq E_k$; then $E_k\in\mathcal{S}_{A_j}\subseteq\mathcal{S}_{A_i}$;
- $\forall k,\ 0\leqslant k<q$, $\{E_k, E_{k+1}\}\in U^j$ (Lemma 2.2); as $A_j\supseteq A_i$, by definition of the order, $j\leqslant i$, then $U^j\subseteq U^i$; therefore, this chain is a chain of $G^i(\mathcal{S}_{A_i})$.

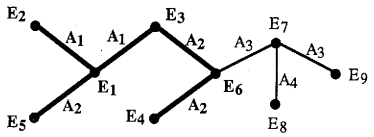


Fig. 6. One of the minimal intergraphs for the hypergraph in Fig. 4. The edges of $G^2(\mathbb{S}_{A_2})$ are shown in bold.

($\Leftarrow$): $\forall E_{i_1}, E_{i_2} \mid E_{i_1} \cap E_{i_2} = A_i$, there is a chain $(E_0 = E_{i_1}, \ldots, E_k, \ldots, E_q = E_{i_2})$ in $G^i(\mathbb{S}_{A_i})$ such that $\forall k, 0 \leqslant k < q$, $A_i \subseteq v_U(\{E_k, E_{k+1}\})$; as $G^i(\mathbb{S}_{A_i})$ is a partial subgraph of G, this chain is also a chain of G. So, $G \in \mathscr{C}(H)$. □

Lemma 2.4. *If $G, G' \in \mathscr{C}(H)$, then $\forall i, 1 \leqslant i < h$, the connected components of $G^i(\mathbb{S}_{A_{i+1}})$ and $G'^i(\mathbb{S}_{A_{i+1}})$ induce the same partition of $\mathbb{S}_{A_{i+1}}$ (see Fig. 7).*

Proof. We need only to prove that E_{i_1}, E_{i_2} connected in $G^i(\mathbb{S}_{A_{i+1}}) \Rightarrow E_{i_1}, E_{i_2}$ connected in $G'^i(\mathbb{S}_{A_{i+1}})$. If E_{i_1} and E_{i_2} are connected in $G^i(\mathbb{S}_{A_{i+1}})$, there is a chain $(E_0 = E_{i_1}, \ldots, E_k, \ldots, E_q = E_{i_2})$ such that $\forall k, 1 \leqslant k < q, \exists r \leqslant i \mid v_U(\{E_k, E_{k+1}\}) = A_r$. Then $E_k, E_{k+1} \in \mathbb{S}_{A_r}$, and, by Lemma 2.3, E_k and E_{k+1} are connected in $G'^r(\mathbb{S}_{A_r})$, and, as $U'^r \subseteq U'^i$ and $\mathbb{S}_{A_r} \subset \mathbb{S}_{A_{i+1}}$, E_k and E_{k+1} are also connected in $G'^i(\mathbb{S}_{A_{i+1}})$. □

Corollary 2.5. *Let $G, G' \in \mathscr{C}(H)$; then $G^i(\mathbb{S}_{A_{i+1}})$ and $G'^i(\mathbb{S}_{A_{i+1}})$ have the same number of connected components. In the following lemma, $p(i)$ denotes this number.*

Lemma 2.6. *Let $G = (\mathscr{E}, U, v_U) \in \mathscr{C}_m(H)$ and $U_{A_i} = \{e \in U \mid v_U(e) = A_i\}$; then $\forall i, 0 \leqslant i < h$, $|U_{A_{i+1}}| = p(i) - 1$.*

Proof. By Lemma 2.3, $G^{i+1}(\mathbb{S}_{A_{i+1}})$ is connected. If $\mathbb{S}_{A_{i+1}}$ is disconnected in G^i, the edges connecting it in G^{i+1} belong to $U_{A_{i+1}}$. The minimal number of such edges is $p(i) - 1$ and each extra edge would be redundant (i.e. its suppression would not disconnect $G^{i+1}(\mathbb{S}_{A_{i+1}})$). As G is minimal, $|U_{A_{i+1}}| = p(i) - 1$. □

Theorem 2.7. *Let $H = (X, \mathscr{E})$ be a hypergraph; if $G = (\mathscr{E}, U, v_U)$ and $G' = (\mathscr{E}, U', v_{U'})$ are two minimal intergraphs of H, then $|U| = |U'|$.*

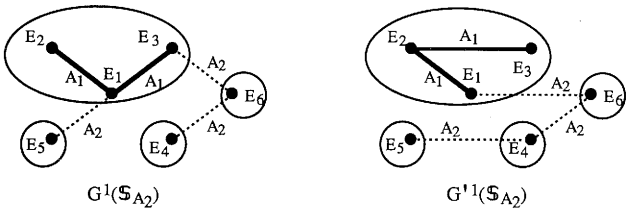


Fig. 7. Illustration of Lemma 2.3 with the hypergraph in Fig. 4 by considering $G^1(\mathbb{S}_{A_2})$.

Proof. By Corollary 2.5 and Lemma 2.6, $\forall i, 1 \leqslant i \leqslant h, |U_{A_i}| = |U'_{A_i}|$; then we have $|U| = |U'|$. □

3. Minimal intergraphs and matroids

Let us recall the basic property of all intergraphs $G(H) = (\mathscr{E}, U, v_U)$ of a hypergraph $H = (X, \mathscr{E})$:

($\mathbb{P}$) $\forall E_i, E_j \in \mathscr{E}$ *if* $E_i \cap E_j \neq \emptyset$, *there exists in* $G(H)$ *a chain* $(E_i = E_1, E_2, \ldots, E_q = E_j)$ *such that* $\forall k, 1 \leqslant k < q, E_i \cap E_j \subseteq v_U(\{E_k, E_{k+1}\})$.

Property $\mathbb{P}$ for intergraphs can be seen as a special connectivity property. In fact, minimal intergraphs are rather similar, w.r.t property $\mathbb{P}$, to spanning trees for graphs. This section links minimal intergraphs and matroids.

Definition 3.1 (Whitney [14]). Let F be a finite set and $\mathscr{F}$ a family of subsets of F; $\mathscr{M} = [\mathrm{F}, \mathscr{F}]$ is a *matroid* on F if and only if

$$\mathscr{F} = \{E \mid E \subseteq B, B \in \mathscr{B}\}$$

with:

(1) $\mathscr{B} \neq \emptyset$ and no element of $\mathscr{B}$ is strictly contained in another element of $\mathscr{B}$;

(2) if $B_1 \in \mathscr{B}$, $B_2 \in \mathscr{B}$, and $e_1 \in B_1$, then there exists $e_2 \in B_2$ s.t. $(B_1 - \{e_1\}) \cup \{e_2\} \in \mathscr{B}$ (*exchange axiom*).

Notations. We use here the same notations as in Section 2; similarly, we take a total order on $\mathscr{I}$ compatible with $\supset$. Let $G = (\mathscr{E}, U, v_U)$ be an intergraph; for $i \in [0, h-1]$, π_i denotes the partition of $\mathbb{S}_{A_{i+1}}$ induced by the connected components of $G^i(\mathbb{S}_{A_{i+1}})$. As usual, if G is a graph and π is a partition of the vertex set of G, G/π denotes the quotient graph.

Lemma 3.2. *Let* $H = (X, \mathscr{E})$ *be a hypergraph; if* $G = (\mathscr{E}, U, v_U) \in \mathscr{C}_{\mathrm{m}}(H)$, *then* $\forall i, 0 \leqslant i < h$, $G^{i+1}(\mathbb{S}_{A_{i+1}})/\pi_i$ *is a tree; moreover, the edges of* $G^{i+1}(\mathbb{S}_{A_{i+1}})/\pi_i$ *are in bijection with* $U_{A_{i+1}}$, *where, as in Lemma* 2.6, $U_{A_{i+1}} = \{e \in U \mid v_U(e) = A_{i+1}\}$ (*see Fig.* 8).

Proof (*by induction on i*). For $i = 0$, π_0 is the trivial partition of $\mathbb{S}_{A_1}$ in isolated vertices; by Lemmas 2.3 and 2.6, $G^1(\mathbb{S}_{A_1})$ is a tree. Suppose property $\mathbb{P}$ holds for $k \leqslant i-1$. By Lemma 2.3, $G^{i+1}(\mathbb{S}_{A_{i+1}})$ is connected. If $G^i(\mathbb{S}_{A_{i+1}})$ is connected, π_i is the trivial partition with only one element. If $\mathbb{S}_{A_{i+1}}$ is disconnected in $G^i(\mathbb{S}_{A_{i+1}})$, the edges connecting it in $G^{i+1}(\mathbb{S}_{A_{i+1}})$ belong to $U_{A_{i+1}}$. By Lemma 2.6, the number of edges connecting the different connected components of $G^i(\mathbb{S}_{A_{i+1}})$ is exactly the number of connected components of $G^i(\mathbb{S}_{A_{i+1}})$ minus 1; so, $G^{i+1}(\mathbb{S}_{A_{i+1}})/\pi_i$ is a tree.

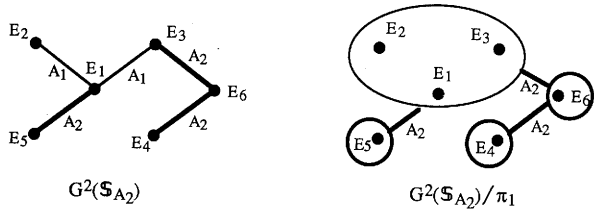


Fig. 8. Lemma 3.2 applied to the hypergraph in Fig. 4. One can notice that $G^{i+1}(\mathbb{S}_{A_{i+1}})/\pi_i$ is a tree and that its edges correspond bijectively to the edges of U_{A_2} (these edges are shown in bold).

To each edge of $G^{i+1}(\mathbb{S}_{A_{i+1}})/\pi_i$ corresponds one edge of $G^{i+1}(\mathbb{S}_{A_{i+1}})$, labelled A_{i+1}, connecting two connected components, and vice versa. Therefore, the edges of $G^{i+1}(\mathbb{S}_{A_{i+1}})/\pi_i$ and $U_{A_{i+1}}$ are in bijection. □

Property 3.3 (Exchange). *Let $H=(X, \mathscr{E})$ be a hypergraph; if $G=(\mathscr{E}, U, v_U)$ and $G'=(\mathscr{E}, U', v_{U'})$ s.t. $G, G' \in \mathscr{C}_m(H)$, then*

$$\forall e \in U, \exists e' \in U' \mid G'' = (\mathscr{E}, (U-\{e\}) \cup \{e'\}) \in \mathscr{C}_m(H).$$

Proof. $\forall e \in U, \exists i, 1 \leqslant i \leqslant h \mid e \in U_{A_i}$. By Lemma 3.2, $G^i(\mathbb{S}_{A_i})/\pi_{i-1}$ and $G'^i(\mathbb{S}_{A_i})/\pi_{i-1}$ are trees; e corresponds to an edge ε of $G^i(\mathbb{S}_{A_i})/\pi_{i-1}$. So, there exists an edge ε' of $G'^i(\mathbb{S}_{A_i})/\pi_{i-1}$ such that the graph constructed from $G^i(\mathbb{S}_{A_i})/\pi_{i-1}$ by suppressing ε and adding ε' is a tree. By Lemma 3.2, the edge ε' corresponds to an edge e' of U'_{A_i}. Therefore, by Lemma 2.3, G'' constructed from G by suppressing e and adding e' is an intergraph. Moreover, as G'' has the same number of edges as the minimal intergraph G, G'' is also minimal. □

In what follows for a given hypergraph $H=(X, \mathscr{E})$, we take:

- F: the edge set of the line graph of H,
- $\mathscr{F}=\{U \mid U \subseteq F$ such that $(\mathscr{E}, U, v_U)$ is a partial graph of a minimal intergraph$\}$,
- $\mathscr{B}=\{B \mid B \subseteq F$ and $(\mathscr{E}, B, v_B) \in \mathscr{C}_m(H)\}$.

Theorem 3.4. *$\mathscr{M}=[F, \mathscr{F}]$ is a matroid on F and $\mathscr{B}$ is the set of bases of $\mathscr{M}$.*

Proof. By using Definition 3.1 for matroids $\mathscr{B}$ is not empty and, by Theorem 2.7, two elements of $\mathscr{B}$ have the same cardinality. The proof of the second part is trivial by Property 3.3. □

It should be noted that Bernstein and Goodman [5] have established implicitly the connections between minimal intergraphs and matroids for a special class of hypergraphs (their minimal intergraphs are trees, which have been called 'join trees' in the terminology of relational databases theory). In fact, Bernstein and Goodman use

Prim's algorithm to compute a join tree of maximal weight. This result is trivial when minimal intergraphs are trees. Theorem 3.4 allows us to extend it to any hypergraph. As a consequence, Janssen et al. [11] give a greedy algorithm to compute the minimal intergraphs of a hypergraph.

Its complexity is $O(n \cdot f + f \cdot a \cdot \log(h) + h \cdot (m + e_m))$, where n is $|X|$, m is $|\mathscr{E}|$, f is the number of edges of the line graph of H, h is $|\mathscr{F}|$, a is $\max(|E_i|)$ for $E_i \in \mathscr{E}$, and e_m is the number of edges of a minimal intergraph of H. This algorithm relies on the matroid structure and is rather similar to Kruskal's algorithm, which computes spanning trees.

4. Minimal intergraphs and cyclicity

For a given hypergraph $H=(X, \mathscr{E})$, a labelled minimal intergraph allows us to reconstruct the hypergraph, with the exception of vertices belonging to only one edge. As the minimal intergraphs of H have the same number of vertices $|\mathscr{E}|$, the same number of connected components p, and the same number of edges, we can take the cyclomatic number of any minimal intergraph as a cyclomatic number of H.

Definition 4.1. Let $H=(X, \mathscr{E})$ be a hypergraph and let $G=(\mathscr{E}, U, v_U) \in \mathscr{C}_m(H)$; the *cyclomatic number* $v(H)$ *of* H is the usual cyclomatic number of G:

$$v(H) = |U| - |\mathscr{E}| + p \quad (p \text{ is the number of connected components of } G).$$

We will now follow with some remarks about connections between this definition and previous works on hypergraph cyclicity, either in combinatorics [1, 4], or in the framework of relational databases theory [2, 6, 8].

Definition 4.2 (Berge [3]). A hypergraph $H=(X, \mathscr{E})$ is *conformal* if and only if the maximal edges of H are the maximal cliques of its two-section $(H)_2$ (the two-section $(H)_2$ of H is the graph with vertex set X, and $\{x, y\}$ is an edge of $(H)_2$ iff there exists an edge of H containing both x and y).

Definition 4.3. Let $H=(X, \mathscr{E})$ be a hypergraph; the following properties are equivalent and define *T-hypergraphs*:

(1) H is conformal and, in every cycle of H with length $\geqslant 3$, there is an edge containing three vertices of this cycle [1, 3].

(2) H is conformal and its 2-section $(H)_2$ is chordal [2, 3].

In the framework of relational databases, T-hypergraphs are known as *α-acyclic hypergraphs.*

Property 4.4 (Beeri et al. [2]). *Let* $H=(X, \mathscr{E})$ *be a hypergraph; if* $G(H) \in \mathscr{C}_m(H)$, *then we have*

$$G(H) \text{ acyclic} \Leftrightarrow H \text{ T-hypergraph}.$$

In fact, this property was expressed as 'H α-acyclic $\Leftrightarrow$ H has a join tree'.

Corollary 4.5. *H is a T-hypergraph $\Leftrightarrow$ $\nu(H)=0$.*

Proof. Trivial by Definition 4.1 and Property 4.4. □

In another context, Acharya and Las Vergnas [1] introduce a parameter μ generalizing the cyclomatic number of graphs to hypergraphs and characterize the hypergraphs H with $\mu(H)=0$ as T-hypergraphs.

Definition 4.6. (Hansen and Las Vergnas [10]). Let $H=(X, \mathscr{E})$ be a hypergraph; the weighted line graph of H is (see Fig. 9) $G_p(H)=(\mathscr{E}, F, p_F)$, with:

- same vertices and same edges as $\mathrm{GR}(H)=(\mathscr{E}, F, v_F)$, the labelled line graph of H,
- p_F is such that $\forall f\in F$, $p_F(f)=|v_F(f)|$.

$p(H)$ denotes the weight of a maximum spanning forest of $G_p(H)$.

Definition 4.7 (Acharya and Las Vergnas [1]). Let $H=(X, \mathscr{E})$ be a hypergraph; the *cyclomatic number of H is the parameter* $\mu(H)$ defined as

$$\mu(H)=\sum_{E_i\in\mathscr{E}}|E_i|-\left|\bigcup_{E_i\in\mathscr{E}}E_i\right|-p(H).$$

Property 4.8 (Acharya and Las Vergnas [1]). *Let $H=(X, \mathscr{E})$ be a hypergraph; then*

$$\mu(H)=0 \Leftrightarrow H \text{ is a T-hypergraph;}$$

Moreover, if H is a graph, $\mu(H)$ is the usual cyclomatic number.

Corollary 4.9. *Let $H=(X, \mathscr{E})$ be a hypergraph; then $\mu(H)=0 \Leftrightarrow \nu(H)=0$.*

In the general case, $\nu(H)\neq\mu(H)$ (see Fig. 10). Nevertheless, there can be other classes of hypergraphs where the equality $\nu(H)=\mu(H)$ holds.

Property 4.10. *If $H=(X, \mathscr{E})$ is a hypergraph such that $\forall E_i\neq E_j\in\mathscr{E}$, $|E_i\cap E_j|\leqslant 1$, then*

$$\nu(H)=\mu(H).$$

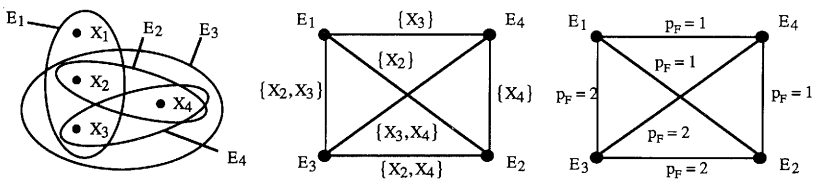


Fig. 9. A hypergraph, its labelled line graph and its weighted line graph.

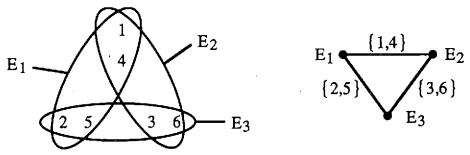


Fig. 10. A hypergraph H and its minimal intergraph $G_m(H)$ with $\nu(H)=1$ and $\mu(H)=2$.

Proof. Let $G=(\mathscr{E}, U, \nu_U)\in\mathscr{G}_m(H)$. Without loss of generality, we consider here connected hypergraphs. In such a case, the number of connected components of G is 1. So, we have $\nu(H)=|U|-|\mathscr{E}|+1$. We also have $|\bigcup_{E_i\in\mathscr{E}} E_i|=|X|$. As $|E_i\cap E_j|\leqslant 1$, $p(H)=|\mathscr{E}|-1$. It is then sufficient to prove that

$$\sum_{E_i\in\mathscr{E}} |E_i|=|X|+|U|.$$

Notations used are the same as in Section 2. For $\mathscr{I}=\{A_1, A_2, \ldots, A_h\}$, we have $\forall i,\ |A_i|=1$ and $\forall i\neq j,\ A_i\cap A_j=\emptyset$. Therefore, the number of edges connecting $\mathbb{S}_{A_i}$ is $|\mathbb{S}_{A_i}|-1$ and, consequently, $|U|=\sum_{1\leqslant i\leqslant h}(|\mathbb{S}_{A_i}|-1)$.

Moreover, if $A_i=\{x_i\}$ and if $d_H(x_i)$ denotes the degree of x_i in H, we have $|\mathbb{S}_{A_i}|=d_H(x_i)$ and, so,

$$\sum_{1\leqslant i\leqslant h}(|\mathbb{S}_{A_i}|-1)=\sum_{1\leqslant i\leqslant h}(d_H(x_i)-1)=\sum_{1\leqslant i\leqslant h} d_H(x_i)-h=|U|.$$

The number of vertices which do not appear in any A_i, i.e. the number of vertices which belong to only one E_j is $|X|-h$; consequently, $\sum_{h<i\leqslant n} d_H(x_i)=|X|-h$.

Therefore,

$$\sum_{1\leqslant i\leqslant h} d_H(x_i)+\sum_{h<i\leqslant n} d_H(x_i)=\sum_{1\leqslant i\leqslant n} d_H(x_i)=|U|+h+|X|-h=|U|+|X|.$$

Since

$$\sum_{1\leqslant i\leqslant n} d_H(x_i)=\sum_{E_i\in\mathscr{E}} |E_i|, \textit{ the equality}$$

$$\sum_{E_i\in\mathscr{E}} |E_i|=|U|+|X|$$

holds. □

Corollary 4.11. *If H is a graph, then $\nu(H)=\mu(H)$, and $\nu(H)$ is the usual cyclomatic number.*

Proof. Trivial by Properties 4.8 and 4.10. □

5. Minimal intergraphs and constraint networks

Constraint networks (or constraint satisfaction problems) [7, 9, 12] involve finding values for variables subject to constraints on which combinations of values are allowed. Examples of constraint networks are map coloring, conjunctive queries in relational databases [2], and understanding line drawings [13].

More precisely, a constraint network is given by:

- a set X of n variables, each variable X_i taking its values in a finite domain D_i;
- a set of m constraints, each constraint involving a subset of the variables E_i; the constraint relation R_i is a subset of the cartesian products of the domains of the variables involved.

In some AI applications such as line drawings understanding, the constraint relations are specified extensionally by the set of tuples satisfying the constraint.

A solution is an assignment of values to variables such that all the constraints are satisfied. For a given constraint network, the problem is either to find all solutions or a solution, or to know if there exists any solution. All these problems are known to be NP-complete.

The pair $(X, \mathscr{E})$ with $\mathscr{E} = \{E_i,\ i = 1, \ldots, m\}$ defines a hypergraph which is the structural component of the constraint network. In binary constraint networks, each constraint involves 2 variables and $(X, \mathscr{E})$ is a graph.

Studies and algorithms on constraint networks deal mainly with binary constraint networks; due to this context, intergraphs can be useful for nonbinary constraint networks in the two following ways [11]:

(1) For a given constraint network CN, with structural part $(X, \mathscr{E})$, they provide an equivalent binary representation CN2 such that:

- the structural component of CN2 $(\mathscr{E}, U)$ is an intergraph of $(X, \mathscr{E})$,
- the domain associated with E_i is R_i,
- for each edge $\{E_i, E_j\}$, R_i and R_j are compatible if and only if they have the same projection on $E_i \cap E_j$.

(2) Intergraphs also allow to give polynomial instances for nonbinary constraint networks. Freuder [9] has shown that for acyclic binary constraint networks there is a greedy algorithm to find a solution. This property can then be extended to nonbinary constraint networks having acyclic minimal intergraphs. Roughly speaking, the tractability of a constraint network is linked to the cyclicity degree of its structural component. In this sense, the cyclomatic number $v(H)$ may be used as an evaluation of this tractability.

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