Induction

Computer Science & Engineering 235: Discrete Mathematics

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Introduction

How can we prove the following quantified statement?

$$\forall s \in S P(x)$$

- For a *finite* set $S = \{s_1, s_2, \ldots, s_n\}$, we can prove that $P(x)$ holds for *each* element because of the equivalence,

$$P(s_1) \wedge P(s_2) \wedge \cdots \wedge P(s_n)$$

- We can use *universal generalization* for infinite sets.
- Another, more sophisticated way is to use *Induction*.

What is Induction?

- If a statement $P(n_0)$ is true for some nonnegative integer; say $n_0 = 1$.
- Also suppose that we are able to prove that *if* $P(k)$ is true for $k \geq n_0$, *then* $P(k+1)$ is also true;

$$P(k) \rightarrow P(k+1)$$

- It follows from these two statements that $P(n)$ is true for all $n \geq n_0$. I.e.

$$\forall n \geq n_0 P(n)$$

This is the basis of the most widely used proof technique; *Induction*.

The Well Ordering Principle I

Why is induction a legitimate proof technique?

At its heart is the *Well Ordering Principle*.

Theorem (Principle of Well Ordering)

Every nonempty set of nonnegative integers has a least element.

Since every such set has a least element, we can form a *base case*.

We can then proceed to establish that the set of integers $n \geq n_0$ such that $P(n)$ is *false* is actually *empty*.

Thus, induction (both "weak" and "strong" forms) are logical equivalences of the well-ordering principle.

Another View I

To look at it another way, assume that the statements

$$P(n_0) \quad (1)$$
$$P(k) \rightarrow P(k+1) \quad (2)$$

are true. We can now use a form of *universal generalization* as follows.

Say we choose an element from the universe of discourse c. We wish to establish that $P(c)$ is true. If $c = n_0$ then we are done.

Another View II

Otherwise, we apply (**??**) above to get

$$\begin{aligned} P(n_0) &\Rightarrow P(n_0+1) \\ &\Rightarrow P(n_0+2) \\ &\Rightarrow P(n_0+3) \\ &\cdots \\ &\Rightarrow P(c-1) \\ &\Rightarrow P(c) \end{aligned}$$

Via a finite number of steps $(c - n_0)$, we get that $P(c)$ is true. Since c was arbitrary, the universal generalization is established.

$$\forall n \geq n_0 P(n)$$

Induction I
Formal Definition

Theorem (Principle of Mathematical Induction)

Given a statement P concerning the integer n, suppose

1. *P is true for some particular integer n_0; $P(n_0) = 1$.*
2. *If P is true for some particular integer $k \geq n_0$ then it is true for $k + 1$.*

Then P is true for all integers $n \geq n_0$; i.e.

$$\forall n \geq n_0 P(n)$$

is true.

Induction II
Formal Definition

- Showing that $P(n_0)$ holds for some initial integer n_0 is called the *Basis Step*. The statement $P(n_0)$ itself is called the *inductive hypothesis*.
- Showing the implication $P(k) \to P(k+1)$ for every $k \geq n_0$ is called the *Induction Step*.
- Together, induction can be expressed as an inference rule.

$$\big(P(n_0) \wedge \forall k \geq n_0 P(k) \to P(k+1)\big) \to \forall n \geq n_0 P(n)$$

Inductive Proofs: Step by Step

1. State and prove the base case
2. State the *Inductive Hypothesis*
3. As an aside, consider where you want to go (identify the *Inductive Conclusion*)
4. Using what you know (Inductive Hypothesis) *prove* the Inductive Conclusion

Example I

Example

Prove that $n^2 \leq 2^n$ for all $n \geq 5$ using induction.

We formalize the statement as $P(n) = (n^2 \leq 2^n)$.

Our *base case* here is for $n = 5$. We directly verify that

$$25 = 5^2 \leq 2^5 = 32$$

and so $P(5)$ is true and thus the induction hypothesis holds.

Example I
Continued

We now perform the induction step and *assume* that $P(k)$ is true. Thus,

$$k^2 \leq 2^k$$

Consider the expression $(k+1)^2$:

$$\begin{array}{rcll}
(k+1)^2 & = & k^2 + 2k + 1 & \\
 & \leq & 2^k + (2k+1) & \text{by the inductive hypothesis} \\
 & \leq & 2^k + (2k+k) & \text{since } k \geq 5 > 1 \\
 & = & 2^k + (3k) & \\
 & \leq & 2^k + 5k & \\
 & \leq & 2^k + k^2 & \text{since } k \geq 5 \\
 & \leq & 2^k + 2^k & \text{by the inductive hypothesis} \\
 & = & 2(2^k) & \\
 & = & 2^{(k+1)} &
\end{array}$$

Example II

Example

Prove that for any $n \geq 1$,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

The base case is easily verified;

$$1 = 1^2 = \frac{(1+1)(2+1)}{6} = 1$$

Now assume that $P(k)$ holds for some $k \geq 1$, so

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Example II
Continued

We want to show that $P(k+1)$ is true; that is, we want to show that

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

However, observe that this sum can be written

$$\sum_{i=1}^{k+1} i^2 = 1^2 + 2^2 + \cdots k^2 + (k+1)^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

Example II
Continued

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad (*) \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
&= \frac{(k+1)\,[k(2k+1)+6(k+1)]}{6} \\
&= \frac{(k+1)\left[2k^2+7k+6\right]}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

Example II
Continued

Thus we have that

$$\sum_{i=1}^{k+1} = \frac{(k+1)(k+2)(2k+3)}{6}$$

so we've established that $P(k) \to P(k+1)$.

Thus, by the principle of mathematical induction,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Example III

Example

Prove that for any integer $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

Define $P(n)$ to be the statement that $3 \mid 2^{2n} - 1$.

Again, we note that the base case is $n = 1$, so we have that

$$2^{2\cdot 1} - 1 = 3$$

which is certainly divisible by 3.

We next assume that $P(k)$ holds. That is, we assume that there exists an integer m such that

$$2^{2k} - 1 = 3m$$

Example III
Continued

Consider $2^{2(k+1)} - 1$:

$$\begin{aligned}
2^{2(k+1)} - 1 &= 4(2^{2k}) - 1 \\
&= 4(3m+1) - 1 \quad \text{by inductive hypothesis} \\
&= 4(3m) + 4 - 1 \\
&= 3(4m) + 3 \\
&= 3(4m+1)
\end{aligned}$$

And we are done, since 3 divides the RHS, it must divide the LHS. Thus, by the principle of mathematical induction, $2^{2n} - 1$ is divisible by 3 for all $n \geq 1$.

Example IV

Example

Prove that $n! > 2^n$ for all $n \geq 4$

The base case holds since $24 = 4! > 2^4 = 16$.

We now make our inductive hypothesis and assume that

$$k! > 2^k$$

for some integer $k \geq 4$

Since $k \geq 4$, it certainly is the case that $k+1 > 2$. Therefore, we have that

$$(k+1)! = (k+1)k! > 2 \cdot 2^k = 2^{k+1}$$

So by the principle of mathematical induction, we have our desired result. $\square$

Example V

Example

Let $m \in \mathbb{Z}$ and suppose that $x \equiv y(\mod m)$. Then for all $n \geq 1$,

$$x^n \equiv y^n(\mod m)$$

The base case here is trivial as it is encompassed by the assumption.

Now assume that it is true for some $k \geq 1$;

$$x^k \equiv y^k(\mod m)$$

Example V

Continued

Since multiplication of corresponding sides of a congruence is still a congruence, we have

$$x \cdot x^k \equiv y \cdot y^k(\mod m)$$

And so

$$x^{k+1} \equiv y^{k+1}(\mod m)$$

□

Example VI

Example

Show that

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$$

for all $n \geq 1$.

The base case is trivial since $1^3 = (1)^2$.

The induction hypothesis will assume that it holds for some $k \geq 1$:

$$\sum_{i=1}^{k} i^3 = \left(\sum_{i=1}^{k} i\right)^2$$

Example VI

Continued

Fact

By another standard induction proof (see the text) the summation of natural numbers up to n is

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

We now consider the summation for $(k+1)$:

$$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$$

Example VI

Continued

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \frac{(k^2(k+1)^2) + 4(k+1)^3}{2^2} \\
&= \frac{(k+1)^2\left[k^2 + 4k + 4\right]}{2^2} \\
&= \frac{(k+1)^2(k+2)^2}{2^2} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2
\end{aligned}$$

So by the PMI, the equality holds. □

Example VII I

Bad Example I

Prove that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- Base case (easy): $n = 1$, then $1 = \frac{1(1+1)}{2}$
- Inductive Hypothesis: $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$
- Inductive Conclusion: $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$
- Observe:

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

$$\left(\sum_{i=1}^{k} i\right) + (k+1) = \frac{(k+1)(k+2)}{2}$$

Example VII II

Bad Example I

$$\left(\sum_{i=1}^{k} i\right) + (k+1) = \frac{(k+1)(k+2)}{2}$$
$$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$
$$\frac{k}{2} + 1 = \frac{(k+2)}{2}$$
$$\frac{k+2}{2} = \frac{(k+2)}{2}$$

- Which is true, so done, right?
- Wrong!: we *started* with the inductive conclusion
- You cannot assume the conclusion: this is *begging the question*

Example VII

Bad Example II

Prove that for all $n \geq 2$,
$$n! < n^n$$

- Base case (easy): $2! = 2 < 4 = 2^2$
- Inductive Hypothesis: $k! < k^k$
- Inductive Conclusion: $(k+1)! < (k+1)^{(k+1)}$
- Observe:
$$(k+1) \cdot k! < (k+1) \cdot (k+1)^{(k)}$$
$$k! < (k+1)^{(k)}$$
$$k^k < (k+1)^{(k)}$$
- Which is true, so done, right?
- Wrong!: we *started* with the inductive conclusion
- You cannot assume the conclusion: this is *begging the question*

Example VII

Bad Example II

Consider this "proof" that all of you will receive the same grade.

Proof.

Let $P(n)$ be the statement that every set of n students receives the same grade. Clearly $P(1)$ is true, so the base case is satisfied.

Now assume that $P(k-1)$ is true. Given a group of k students, apply $P(k-1)$ to the subset $\{s_1, s_2, \ldots s_{k-1}\}$. Now, separately apply the induction hypothesis to the subset $\{s_2, s_3, \ldots, s_k\}$.

Combining these two facts tells us that $P(k)$ is true. Thus, $P(n)$ is true for all students. □

Example VII

The *Bad* Example - Continued

- The mistake is not the base case, $P(1)$ is true.
- Also, it *is* the case that, say $P(73) \to P(74)$, so this cannot be the mistake.

The error is in $P(1) \to P(2)$ which is certainly not true; we cannot combine the two inductive hypotheses to get $P(2)$.

Strong Induction I

Another form of induction is called the "strong form".

Despite the name, it is *not* a *stronger* proof technique.

In fact, we have the following.

Lemma

The following are equivalent.

- *The Well Ordering Principle*
- *The Principle of Mathematical Induction*
- *The Principle of Mathematical Induction, Strong Form*

Strong Induction II

Theorem (Principle of Mathematical Induction (Strong Form))

Given a statement P concerning the integer n, suppose

1. *P is true for some particular integer n_0; $P(n_0) = 1$.*
2. *If $k > n_0$ is any integer and P is true for all integers l in the range $n_0 \leq l < k$, then it is true also for k.*

Then P is true for all integers $n \geq n_0$; i.e.
$$\forall (n \geq n_0) P(n)$$
is true.

Strong Form Example

Fundamental Theorem of Arithmetic

Recall that the Fundamental Theorem of Arithmetic states that any integer $n \geq 2$ can be written as a unique product of primes.

We'll use the strong form of induction to prove this.

Let $P(n)$ be the statement "n can be written as a product of primes."

Clearly, $P(2)$ is true since 2 is a prime itself. Thus the base case holds.

Strong Form Example

Fundamental Theorem of Arithmetic - Continued

We make our inductive hypothesis. Here we assume that the predicate P holds for *all* integers less than some integer $k \geq 2$; i.e. we assume that

$$P(2) \wedge P(3) \wedge \cdots \wedge P(k)$$

is true.

We want to show that this implies $P(k+1)$ holds. We consider two cases.

If $k+1$ is prime, then $P(k+1)$ holds and we are done.

Else, $k+1$ is a composite and so it has factors u, v such that $2 \leq u, v < k+1$ such that

$$u \cdot v = k+1$$

Strong Form Example

Fundamental Theorem of Arithmetic - Continued

We now apply the inductive hypothesis; both u and v are less than $k+1$ so they can both be written as a unique product of primes;

$$u = \prod_i p_i, \quad v = \prod_j p_j$$

Therefore,

$$k+1 = \left(\prod_i p_i\right)\left(\prod_j p_j\right)$$

and so by the strong form of the PMI, $P(k+1)$ holds. $\square$

Strong Form Example

GCD

Recall the following.

Lemma

If $a, b \in \mathbb{N}$ are such that $\gcd(a,b) = 1$ then there are integers s, t such that

$$gcd(a,b) = 1 = sa + tb$$

We will prove this using the strong form of induction.

Strong Form Example

GCD

Let $P(n)$ be the statement

$$a, b \in \mathbb{N} \wedge \gcd(a,b) = 1 \wedge a + b = n \Rightarrow \exists s, t \in \mathbb{Z}, as + tb = 1$$

Our base case here is when $n = 2$ since $a = b = 1$.

For $s = 1, t = 0$, the statement $P(2)$ is satisfied since

$$st + bt = 1 \cdot 1 + 1 \cdot 0 = 1$$

Strong Form Example

GCD

We now form the inductive hypothesis. Suppose $n \in \mathbb{N}, n \geq 2$ and assume that $P(k)$ is true for all k with $2 \leq k \leq n$.

Now suppose that for $a, b \in \mathbb{N}$,

$$\gcd(a,b) = 1 \wedge a + b = n + 1$$

We consider three cases.

Strong Form Example
GCD

Case 1 $a = b$

In this case

$$\begin{aligned} \gcd(a, b) &= \gcd(a, a) && \text{by definition} \\ &= a && \text{by definition} \\ &= 1 && \text{by assumption} \end{aligned}$$

Therefore, since the $\gcd$ is one, it must be the case that $a = b = 1$ and so we simply have the base case, $P(2)$.

Strong Form Example
GCD

Case 2 $a < b$

Since $b > a$, it follows that $b - a > 0$ and so

$$\gcd(a, b) = \gcd(a, b - a) = 1$$

(Why?)

Furthermore,

$$2 \leq a + (b - a) = n + 1 - a \leq n$$

Strong Form Example
GCD

Since $a + (b - a) \leq n$, we can apply the inductive hypothesis and conclude that $P(n + 1 - a) = P(a + (b - a))$ is true.

This implies that there exist integers s_0, t_0 such that

$$as_0 + (b - a)t_0 = 1$$

and so

$$a(s_0 - t_0) + bt_0 = 1$$

So for $s = s_0 - t_0$ and $t = t_0$ we get

$$as + bt = 1$$

Thus, $P(n + 1)$ is established for this case.

Strong Form Example
GCD

Case 3 $a > b$ This is completely symmetric to case 2; we use $a - b$ instead of $b - a$.

Since all three cases handle every possibility, we've established that $P(n + 1)$ is true and so by the strong PMI, the lemma holds. $\square$