Relations

Computer Science & Engineering 235: Discrete Mathematics

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Introduction

Recall that a relation between elements of two sets is a subset of their Cartesian product (of ordered pairs).

Definition

A binary relation from a set $A$ to a set $B$ is a subset

$$R \subseteq A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Note the difference between a relation and a function: in a relation, each $a \in A$ can map to multiple elements in $B$. Thus, relations are generalizations of functions.

If an ordered pair $(a, b) \in R$ then we say that $a$ is related to $b$. We may also use the notation $aRb$ and $a \not R b$.

Relations

To represent a relation, you can enumerate every element in $R$.

Example

Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_1, b_2, b_3\}$ let $R$ be a relation from $A$ to $B$ as follows:

$$R = \{(a_1, b_1), (a_1, b_2), (a_2, b_2), (a_3, b_1), (a_3, b_2), (a_3, b_3), (a_5, b_1)\}$$

You can also represent this relation graphically.

Figure: Graphical Representation of a Relation

Reflexivity

Definition

There are several properties of relations that we will look at. If the ordered pairs $(a, a) \in R$ for every $a \in A$ then it is called reflexive.

Definition

A relation $R$ on a set $A$ is called reflexive if

$$\forall a \in A ((a, a) \in R)$$

Exercise: Give some examples of ordered pairs $(a, b) \in \mathbb{N}^2$ that are not in each of these relations.
Reflexivity

Example

Recall the following relations; which is reflexive?

\[
R_1 = \{(a, b) \mid a \leq b\} \\
R_2 = \{(a, b) \mid a, b \in \mathbb{N}, \frac{a}{b} \in \mathbb{Z}\} \\
R_3 = \{(a, b) \mid a, b \in \mathbb{N}, a - b = 2\}
\]

- \(R_1\) is reflexive since for every \(a \in \mathbb{N}\), \(a \leq a\).
- \(R_2\) is not reflexive since \(\frac{a}{b}\) is undefined (not an integer).
- Though all other elements are reflexive.
- \(R_3\) is not reflexive since \(a - a = 0\) for every \(a \in \mathbb{N}\).

Symmetry II

Definition

Some things to note:

- A symmetric relationship is one in which if \(a\) is related to \(b\) then \(b\) must be related to \(a\).
- An antisymmetric relationship is similar, but such relations hold only when \(a = b\).
- An antisymmetric relationship is not necessarily a reflexive relationship.
- A relation can be both symmetric and antisymmetric or neither or have one property but not the other!
- A relation that is not symmetric is not asymmetric.

Symmetric Relations

Example

Let \(R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}\). Is \(R\) reflexive? Symmetric? Antisymmetric?

- It is clearly not reflexive since for example \((2, 2) \notin R\).
- It is symmetric since \(x^2 + y^2 = y^2 + x^2\) (i.e. addition is commutative).
- It is not antisymmetric since \((\frac{1}{7}, \sqrt{2}) \in R\) and \((\sqrt{2}, \frac{1}{7}) \in R\) but \(\frac{1}{7} \neq \sqrt{2}\).

Symmetry I

Definition

A relation \(R\) on a set \(A\) is called symmetric if

\[(b, a) \in R \iff (a, b) \in R\]

for all \(a, b \in A\).

A relation \(R\) on a set \(A\) is called antisymmetric if

\[\forall a, b, \quad ((a, b) \in R \wedge (b, a) \in R) \rightarrow a = b\]

for all \(a, b \in A\).

Transitivity

Definition

A relation \(R\) on a set \(A\) is called transitive if whenever \((a, b) \in R\) and \((b, c) \in R\) then \((a, c) \in R\) for all \(a, b, c \in A\). Equivalently,

\[\forall a, b, c \in A, (aRb \wedge bRc) \rightarrow aRc\]
**Transitivity**

**Examples**

<table>
<thead>
<tr>
<th>Example</th>
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<tbody>
<tr>
<td>Is the relation $R = {(x, y) \in \mathbb{R}^2 \mid x \leq y}$ transitive?</td>
</tr>
<tr>
<td>Yes it is transitive since $(x \leq y) \land (y \leq z) \Rightarrow x \leq z$.</td>
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<tr>
<td>Is the relation $R = {(a, b), (b, a), (a, a)}$ transitive?</td>
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<tr>
<td>No since $bRa$ and $aRb$ but $b \not\in R$.</td>
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<tr>
<th>Other Properties</th>
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<tbody>
<tr>
<td>▶ A relation is asymmetric if $\forall a, b \in R \rightarrow (b, a) \not\in R$</td>
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<tr>
<td>▶ A relation is irreflexive if $\forall a \in R \rightarrow (a, a) \not\in R$</td>
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<tr>
<th>Lemma</th>
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<td>A relation $R$ on a set $A$ is asymmetric if and only if $R$ is irreflexive and antisymmetric.</td>
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<tr>
<th>Combining Relations</th>
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<tr>
<td>Example</td>
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<tr>
<td>Let $A = {1, 2, 3, 4}$, $B = {1, 2, 3}$, $R_1 = {(1, 2), (1, 3), (1, 4), (2, 2), (3, 4), (4, 1), (4, 2)}$, $R_2 = {(1, 1), (1, 2), (1, 3), (2, 3)}$.</td>
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<tr>
<td>$R_1 \cup R_2 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (3, 4), (4, 1), (4, 2)}$</td>
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<tr>
<td>$R_1 \cap R_2 = {(1, 2), (1, 3)}$</td>
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<tr>
<td>$R_1 \setminus R_2 = {(1, 4), (2, 2), (3, 4), (4, 1), (4, 2)}$</td>
</tr>
<tr>
<td>$R_2 \setminus R_1 = {(1, 1), (2, 3)}$</td>
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**Combining Relations**

Relations are simply sets, that is subsets of ordered pairs of the Cartesian product of a set.

It therefore makes sense to use the usual set operations, intersection $\cap$, union $\cup$ and set difference $A \setminus B$ to combine relations to create new relations.

Sometimes combining relations endows them with the properties previously discussed. For example, two relations may not be transitive alone, but their union may be.

**Relation Composition**

**Definition**

Let $R_1$ be a relation from the set $A$ to $B$ and $R_2$ be a relation from $B$ to $C$. i.e. $R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$. The composite of $R$ and $S$ is the relation consisting of ordered pairs $(a, c)$ where $a \in A$, $c \in C$ and for which there exist $b \in B$ such that $(a, b) \in R_1$ and $(b, c) \in R_2$. We denote the composite of $R_1$ and $R_2$ by $R_1 \circ R_2$. 

$$R_1 \circ R_2$$
Relation Composition I

Example

Construct $R_1 \circ R_1$:

- $(a_1, a_1) \in R_1 \circ R_1$? (yes, with $b = a_4$)
- $(a_1, a_4) \in R_1 \circ R_1$? (yes, with $b = a_3$)
- $(a_2, a_1) \in R_1 \circ R_1$? (no)

Relation Composition II

Example

$$
\begin{array}{c}
0_1 \\
\vdots \\
0_4 \\
0_4 \\
\vdots \\
0_1
\end{array}
$$

$R \circ R$

A

A

A

Figure: Composition of a relation with itself.

Powers of Relations

Using this composite way of combining relations (similar to function composition) allows us to recursively define powers of a relation $R$.

Definition

Let $R$ be a relation on $A$. The powers, $R^n$, $n = 1, 2, 3, \ldots$, are defined recursively by

$$
\begin{align*}
R^1 & = R \\
R^{n+1} & = R^n \circ R
\end{align*}
$$

Powers of Relations

The powers of relations give us a nice characterization of transitivity.

Theorem

A relation $R$ is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

Representing Relations

We have seen ways of graphically representing a function/relation between two (different) sets—specifically a graph with arrows between nodes that are related.

We will look at two alternative ways of representing relations; 0-1 matrices and directed graphs.
0-1 Matrices I

A 0-1 matrix is a matrix whose entries are either 0 or 1.

Let R be a relation from \( A = \{a_1, a_2, \ldots, a_n\} \) to \( B = \{b_1, b_2, \ldots, b_m\} \).

Note that we have induced an ordering on the elements in each set. Though this ordering is arbitrary, it is important to be consistent; that is, once we fix an ordering, we stick with it.

In the case that \( A = B \), R is a relation on A, and we choose the same ordering.

0-1 Matrices II

The relation R can therefore be represented by a \((n \times m)\) sized 0-1 matrix \( M_R = [m_{i,j}] \) as follows.

\[
m_{i,j} = \begin{cases} 
1 & \text{if } (a_i, b_j) \in R \\
0 & \text{if } (a_i, b_j) \notin R
\end{cases}
\]

Intuitively, the \((i,j)\)-th entry is 1 if and only if \( a_i \in A \) is related to \( b_j \in B \).

0-1 Matrices III

An important note: the choice of row or column-major form is important. The \((i,j)\)-th entry refers to the \(i\)-th row and \(j\)-th column. The size, \((n \times m)\), refers to the fact that \( M_R \) has \( n \) rows and \( m \) columns.

Though the choice is arbitrary, switching between row-major and column-major is a bad idea, since for \( A \neq B \), the Cartesian products \( A \times B \) and \( B \times A \) are not the same.

In matrix terms, the transpose, \((M_R)^T\), does not give the same relation. This point is moot for \( A = B \).

0-1 Matrices IV

Let \( A = \{a_1, a_2, a_3, a_4, a_5\} \) and \( B = \{b_1, b_2, b_3\} \) let R be a relation from \( A \) to \( B \) as follows:

\[
R = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_2, b_1), (a_3, b_1), (a_3, b_2), (a_5, b_1)\}
\]

What is \( M_R \)?

Clearly, we have a \((5 \times 3)\) sized matrix.

\[
M_R = \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
1 & 1 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

Matrix Representations

Useful Characteristics

A 0-1 matrix representation makes checking whether a relation is reflexive, symmetric and transitive or not very easy.

**Reflexivity** – For \( R \) to be reflexive, \( \forall (a,a) \in R \). By the definition of the 0-1 matrix, \( R \) is reflexive if and only if \( m_{i,i} = 1 \) for \( i = 1, 2, \ldots, n \). Thus, one simply has to check the diagonal.
Matrix Representations

Useful Characteristics

Symmetry – R is symmetric if and only if for all pairs \((a, b)\), \(aRb \Rightarrow bRa\). In our defined matrix, this is equivalent to \(m_{i,j} = m_{j,i}\) for every pair \(i, j = 1, 2, \ldots, n\).

Alternatively, R is symmetric if and only if \(M_R = (M_R)^T\).

Antisymmetry – To check antisymmetry, you can use a disjunction; that is R is antisymmetric if \(m_{i,j} = 1\) with \(i \neq j\) then \(m_{j,i} = 0\). Thus, for all \(i, j = 1, 2, \ldots, n\), \(i \neq j\), \((m_{i,j} = 0) \lor (m_{j,i} = 0)\).

What is a simpler logical equivalence?

\[\forall i, j= 1, 2, \ldots, n; i \neq j \left(\neg (m_{i,j} \land m_{j,i})\right)\]

Matrix Representations

Combining Relations

Combining relations is also simple—union and intersection of relations is nothing more than entry-wise boolean operations.

Union – An entry in the matrix of the union of two relations \(R_1 \cup R_2\) is 1 if and only if at least one of the corresponding entries in \(R_1\) or \(R_2\) is one. Thus

\[M_{R_1 \cup R_2} = M_{R_1} \lor M_{R_2}\]

Intersection – An entry in the matrix of the intersection of two relations \(R_1 \cap R_2\) is 1 if and only if both of the corresponding entries in \(R_1\) and \(R_2\) is one. Thus

\[M_{R_1 \cap R_2} = M_{R_1} \land M_{R_2}\]

Matrix Representations

Composite Relations

One can also compose relations easily with 0-1 matrices. We will not discuss how here, rather please read this for yourself.

You will need to read section 2.7 for some definitions (Boolean product of matrices).

Remember that recursively composing a relation \(R^n\), \(n = 1, 2, \ldots\) gives a nice characterization of transitivity.

Using these ideas, you can also determine if a relation is transitive or not by computing the transitive closure (discussed in the next section).

Directed Graphs

We will get more into graphs later on, but we briefly introduce them here since they can be used to represent relations.

In the general case, we’ve already seen directed graphs used to represent relations. However, for relations on a set \(A\), it makes more sense to use a general graph rather than have two copies of the set in the diagram.

Matrix Representations

Example

\[M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}\]

Is \(R\) reflexive? Symmetric? Antisymmetric?

▶ Clearly it is not reflexive since \(m_{2,2} = 0\).
▶ It is not symmetric either since \(m_{2,1} \neq m_{1,2}\).
▶ It is, however, antisymmetric. You can verify this for yourself.

Matrix Representations

Example

Let

\[M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, M_{R_2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\]

What is \(M_{R_1 \cup R_2}\) and \(M_{R_1 \cap R_2}\)?

\[M_{R_1 \cup R_2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, M_{R_1 \cap R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\]

How does combining the relations change their properties?
Directed Graphs I

**Definition**

A directed graph (or digraph) consists of a set $V$ of vertices (or nodes) together with a set $E$ of edges of ordered pairs of elements of $V$. We write $G = (V, E)$.

Directed Graphs II

**Example**

Directed Graphs III

Let $A = \{a_1, a_2, a_3, a_4\}$ and let $R$ be a relation on $A$ defined as:

$$R = \{(a_1, a_2), (a_1, a_3), (a_1, a_4), (a_2, a_3), (a_2, a_4), (a_3, a_1), (a_3, a_4), (a_4, a_3), (a_4, a_4)\}$$

Directed Graph Representation I

**Usefulness**

Again, a directed graph offers some insight as to the properties of a relation.

**Reflexivity** – In a digraph, a relation is reflexive if and only if every vertex has a self loop.

**Symmetry** – In a digraph, a represented relation is symmetric if and only if for every edge from $x$ to $y$ there is also a corresponding edge from $y$ to $x$.

Directed Graph Representation II

**Usefulness**

**Antisymmetry** – A represented relation is antisymmetric if and only if there is never a back edge for each directed edge between distinct vertices.

**Transitivity** – A digraph is transitive if for every pair of edges $(x, y)$ and $(y, z)$ there is also a directed edge $(x, z)$ (though this may be harder to verify in more complex graphs visually).

Closures

**Definition**

If a given relation $R$ is not reflexive (or symmetric, antisymmetric, transitive) can we transform it into a relation $R'$ that is?

**Example**

Let $R = \{(1, 2), (2, 1), (2, 2), (3, 1), (3, 3)\}$ is not reflexive. How can we make it reflexive?

In general, we’d like to change the relation as little as possible. To make this relation reflexive we simply have to add $(1, 1)$ to the set.

Inducing a property on a relation is called its closure. In the example, $R'$ is the reflexive closure.
Warshall’s Algorithm

Key Ideas

In any set \( A \) with \( |A| = n \) elements, any transitive relation will be built from a sequence of relations that has a length at most \( n \).

Why? Consider the case where \( A \) contains the relations

\[(a_1, a_2), (a_2, a_3), \ldots, (a_{n-1}, a_n)\]

Then \( (a_1, a_n) \) is required to be in \( A \) for \( A \) to be transitive.

Thus, by the previous theorem, it suffices to compute (at most) \( R^n \). Recall that \( R^k = R \circ R^{k-1} \) is calculated using a Boolean matrix product. This gives rise to a natural algorithm.

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Closures I

In general, the reflexive closure of a relation \( R \) on \( A \) is \( R \cup \Delta \) where \( \Delta = \{(a, a) \mid a \in A\} \) is the diagonal relation on \( A \).

Question: How can we compute the reflexive closure using a 0-1 matrix representation? Digraph representation?

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Closures II

Also, transitive closures can be made using a previous theorem:

**Theorem**

A relation \( R \) is transitive if and only if \( R^n \subseteq R \) for \( n = 1, 2, 3, \ldots \).

Thus, if we can compute \( R^k \) such that \( R^k \subseteq R^n \) for all \( n \geq k \), then \( R^k \) is the transitive closure.

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Warshall’s Algorithm

**Example**

Compute the transitive closure of the relation

\[ R = \{(1, 1), (1, 2), (1, 4), (2, 2), (2, 3), (3, 1), (3, 4), (4, 1), (4, 4)\} \]

on \( A = \{1, 2, 3, 4\} \)
Equivalence Relations

Consider the set of every person in the world. Now consider a relation such that \((a, b) \in R\) if \(a\) and \(b\) are siblings.

Clearly, this relation is reflexive, symmetric and transitive. Such a unique relation is called an equivalence relation.

Definition

A relation on a set \(A\) is an equivalence relation if it is reflexive, symmetric and transitive.

Equivalence Classes II

Elements in \([a]_R\) are called representatives of the equivalence class.

Theorem

Let \(R\) be an equivalence relation on a set \(A\. TFAE:

1. \(a R b\)
2. \([a] = [b]\)
3. \([a] \cap [b] \neq \emptyset\)

Partitions I

Equivalence classes are important because they can partition a set \(A\) into disjoint non-empty subsets \(A_1, A_2, \ldots, A_l\) where each equivalence class is, in some sense, self-contained.

Note that a partition satisfies these properties:

- \(\bigcup_{i=1}^l A_i = A\)
- \(A_i \cap A_j = \emptyset\) for \(i \neq j\)
- \(A_i \neq \emptyset\) for all \(i\)

Partitions II

For example, if \(R\) is a relation such that \((a, b) \in R\) if \(a\) and \(b\) live in the same US state (or outside the US), then \(R\) is an equivalence relation that partitions US residents into 50 equivalence classes.

Theorem

Let \(R\) be an equivalence relation on a set \(S\). Then the equivalence classes of \(R\) form a partition of \(S\). Conversely, given a partition \(A_i\) of the set \(S\), there is an equivalence relation \(R\) that has the sets \(A_i\) as its equivalence classes.

Visual Interpretation

In a 0-1 matrix, if the elements are ordered into their equivalence classes, equivalence classes/partitions form perfect squares of 1s (and zeros elsewhere).

In a digraph, equivalence classes form a collection of disjoint complete graphs.

Example

Say that we have \(A = \{1, 2, 3, 4, 5, 6, 7\}\) and \(R\) is an equivalence relation that partitions \(A\) into \(A_1 = \{1, 2\}, A_2 = \{3, 4, 5, 6\}\) and \(A_3 = \{7\}\). What does the 0-1 matrix look like? Digraph?
Equivalence Relations

Example I
Example
Let \( R = \{(a, b) \mid a, b \in \mathbb{R}, a \leq b\} \)

- Reflexive?
- Transitive?
- Symmetric? No, it is not since, in particular, \( 4 \leq 5 \) but \( 5 \not\leq 4 \).
- Thus, \( R \) is not an equivalence relation.

Equivalence Relations

Example II
Example
Let \( R = \{(a, b) \mid a, b \in \mathbb{Z}, a = b\} \)

- Reflexive?
- Transitive?
- Symmetric?
- What are the equivalence classes that partition \( \mathbb{Z} \)?

Equivalence Relations

Example III
Example
For \((x, y), (u, v) \in \mathbb{R}^2\) define
\[
R = \{(x, y), (u, v) \mid x^2 + y^2 = u^2 + v^2\}
\]
Show that \( R \) is an equivalence relation. What are the equivalence classes it defines (i.e. what are the partitions of \( \mathbb{R} \))?

Equivalence Relations

Example IV
Example
Given \( n, r \in \mathbb{N} \), define the set
\[
n\mathbb{Z} + r = \{na + r \mid a \in \mathbb{Z}\}
\]
- For \( n = 2, r = 0 \), \( 2\mathbb{Z} \) represents the equivalence class of all even integers.
- What \( n, r \) give the equivalence class of all odd integers?
- If we set \( n = 3, r = 0 \) we get the equivalence class of all integers divisible by 3.
- If we set \( n = 3, r = 1 \) we get the equivalence class of all integers divisible by 3 with a remainder of one.
- In general, this relation defines equivalence classes that are, in fact, congruence classes. (see chapter 2, to be covered later).