Combinatorics

Computer Science & Engineering 235: Discrete Mathematics

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Combinatorics I

Introduction

Combinatorics is the study of collections of objects. Specifically, counting objects, arrangement, derangement, etc. of objects along with their mathematical properties.

Counting objects is important in order to analyze algorithms and compute discrete probabilities.

Originally, combinatorics was motivated by gambling: counting configurations is essential to elementary probability.

In addition, combinatorics can be used as a proof technique. A combinatorial proof is a proof method that uses counting arguments to prove a statement.

Combinatorics I

Motivating Example

How many arrangements are there of a deck of 52 cards?

The standard deck (The Mameluke deck) is thought to be 1000 years old. Have all possible 52! been dealt?

Suppose that 5 billion people have dealt 1 hand every second for the last 1000 years. Percentage of deals that have occurred:

\[
5 \times 10^9 \cdot 1000 \cdot 365.25 \cdot 24 \cdot 60 \cdot 60 \approx 1.9562 \times 10^{-48}
\]

To even deal 1% of all hands, we would require 5.11 \times 10^{48} years (quindecillion).

Combinations

Choosing \(k\) elements from a set of cardinality \(n\) is a combination

Notations:

\[
C_{n}^{k} = C(n, k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}
\]

Combinations are unordered

Common usage: choosing singletons: \(\binom{n}{1} = n\)

Choosing pairs: \(\binom{n}{2} = \frac{n(n-1)}{2}\)

Permutations

Arranging \(n\) elements is a permutation

Number of permutations: \(n!\)

Permutations are ordered

Product Rule

If two events are not mutually exclusive (that is, we do them separately), then we apply the product rule.

Theorem (Product Rule)

Suppose a procedure can be accomplished with two disjoint subtasks. If there are \(n_1\) ways of doing the first task and \(n_2\) ways of doing the second, then there are

\[n_1 \cdot n_2\]

ways of doing the overall procedure.
**Sum Rule I**

If two events are mutually exclusive, that is, they cannot be done at the same time, then we must apply the sum rule.

*Theorem (Sum Rule)*

If an event $e_1$ can be done in $n_1$ ways and an event $e_2$ can be done in $n_2$ ways and $e_1$ and $e_2$ are mutually exclusive, then the number of ways of both events occurring is $n_1 + n_2$.

**Sum Rule II**

There is a natural generalization to any sequence of $m$ tasks; namely the number of ways $m$ mutually exclusive events can occur is

$$n_1 + n_2 + \cdots + n_{m-1} + n_m$$

We can give another formulation in terms of sets. Let $A_1, A_2, \ldots, A_m$ be pairwise disjoint sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

In fact, this is a special case of the general Principle of Inclusion-Exclusion.

**Principle of Inclusion-Exclusion (PIE) I**

*Theorem*

Let $A_1, A_2, \ldots, A_n$ be finite sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n|$$

**Principle of Inclusion-Exclusion (PIE) III**

*Introduction*

More generally, we have the following.

*Lemma*

Let $A, B$ be subsets of a finite set $U$. Then

1. $|A \cup B| = |A| + |B| - |A \cap B|$
2. $|A \cap B| \leq \min\{|A|, |B|\}$
3. $|A \setminus B| = |A| - |A \cap B| \geq |A| - |B|$
4. $|\overline{A}| = |U| - |A|$
5. $|A \oplus B| = |A\cup B| - |A\cap B| = A + B - 2|A \cap B| = |A\setminus B| + |B\setminus A|$
6. $|A \times B| = |A| \cdot |B|$
Principle of Inclusion-Exclusion (PIE) II
Example I

Each summation is over all $i$, pairs $i, j$ with $i < j$, triples $i, j, k$ with $i < j < k$ etc.

Principle of Inclusion-Exclusion (PIE) III
Theorem

To illustrate, when $n = 3$, we have

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - \left[|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\right] + \left[|A_1 \cap A_2 \cap A_3|\right]$$

Principle of Inclusion-Exclusion (PIE) IV
Theorem

To illustrate, when $n = 4$, we have

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| - \left[|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|\right] + \left[|A_1 \cap A_2 \cap A_3 + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|\right] - \left[|A_1 \cap A_2 \cap A_3 \cap A_4|\right]$$

Principle of Inclusion-Exclusion (PIE) I
Example I

How many integers between 1 and 300 (inclusive) are

1. ... Divisible by 5 but by neither 3 nor 7?
2. Divisible by 3 and by 5 but not by 7?
3. Divisible by 5 but by neither 3 nor 7?

Let

- $A = \{n \mid 1 \leq n \leq 300 \land 3 \mid n\}$
- $B = \{n \mid 1 \leq n \leq 300 \land 5 \mid n\}$
- $C = \{n \mid 1 \leq n \leq 300 \land 7 \mid n\}$

By the principle of inclusion-exclusion, we have that

$$|A \cup B \cup C| = |A| + |B| + |C| - \left[|A \cap B| + |A \cap C| + |B \cap C|\right] + \left[|A \cap B \cap C|\right]$$

It remains to find the final 4 cardinalities. All three divisors, 3, 5, 7 are relatively prime. Thus, any integer that is divisible by both 3 and 5 must simply be divisible by 15.

Principle of Inclusion-Exclusion (PIE) II
Example I

How big are each of these sets? We can easily use the floor function:

- $|A| = \left\lceil \frac{300}{3} \right\rceil = 100$
- $|B| = \left\lceil \frac{300}{5} \right\rceil = 60$
- $|C| = \left\lceil \frac{300}{7} \right\rceil = 42$

For (1) above, we are asked to find $|A \cup B \cup C|$.


**Principle of Inclusion-Exclusion (PIE) IV**

Example I

Using the same reasoning for all pairs (and the triple) we have

\[
|A \cap B| = \left\lfloor \frac{300}{15} \right\rfloor = 20 \\
|A \cap C| = \left\lfloor \frac{300}{21} \right\rfloor = 14 \\
|B \cap C| = \left\lfloor \frac{300}{35} \right\rfloor = 8 \\
|A \cap B \cap C| = \left\lfloor \frac{300}{105} \right\rfloor = 2
\]

Therefore,

\[
|A \cup B \cup C| = 100 + 60 + 42 - 20 - 14 - 8 + 2 = 162
\]

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**Principle of Inclusion-Exclusion (PIE) VI**

Example I

For (3) above, we are asked to find

\[
|B \setminus (A \cup C)| = |B| - |B \cap (A \cup C)| \\
\]

By distributing $B$ over the intersection, we get

\[
|B \cap (A \cup C)| = |(B \cap A) \cup (B \cap C)| \\
= |B \cap A| + |B \cap C| - |(B \cap A) \cap (B \cap C)| \\
= |B \cap A| + |B \cap C| - |B \cap A \cap C| \\
= 20 + 8 - 2 = 26
\]

So the answer is $|B| - 26 = 60 - 26 = 34$.

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**Principle of Inclusion-Exclusion (PIE) II**

Example II

▶ There are $n$ such choices:

\[
\binom{n}{1}
\]

▶ Thus, the number of functions that do not map to (at least) a single element is:

\[
\binom{n}{1}(n-1)^m
\]

▶ We’ve over counted though: when we exclude $b_2$, then we are recounting functions that also exclude $b_1$

▶ Need to restore counts: consider pairs

▶ Consider all functions such that no element maps to a pair of elements:

\[
\binom{n}{2}(n-2)^m
\]

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**Principle of Inclusion-Exclusion (PIE) V**

Example I

For (2) above, it is enough to find

\[
|(A \cap B) \setminus C|
\]

By the definition of set-minus,

\[
|(A \cap B) \setminus C| = |A \cap B| - |A \cap B \cap C| = 20 - 2 = 18
\]

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**Principle of Inclusion-Exclusion (PIE) I**

Example II

The principle of inclusion-exclusion can be used to count the number of onto functions.

▶ Let $A = \{a_1, a_2, \ldots, a_m\}, |A| = m$

▶ Let $B = \{b_1, b_2, \ldots, b_n\}, |B| = n$

▶ Say $m \geq n$ (otherwise no onto functions exist)

▶ Observe: total number of functions is $n^m$

▶ Consider all functions such that no element maps to $b_1$:

\[
(n-1)^m
\]

▶ Generalize this: consider all functions such that no element maps to $b_i$ for a particular $i$
Principle of Inclusion-Exclusion (PIE) IV

Example II

Let $A, B$ be non-empty sets of cardinality $m, n$ with $m \geq n$. Then there are

$$\sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^m$$

i.e.

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^{n-1}\binom{n}{n-1}1^m$$

onto functions $f : A \to B$. 

Principle of Inclusion-Exclusion (PIE) VI

Example II

How many ways of giving out 6 pieces of candy to 3 children if each child must receive at least one piece?

This can be modeled by letting $A$ represent the set of candies and $B$ be the set of children.

Then a function $f: A \to B$ can be interpreted as giving candy $a_i$ to child $c_j$.

Since each child must receive at least one candy, we are considering only onto functions.

Principle of Inclusion-Exclusion (PIE) VII

Example II

To count how many there are, we apply the theorem and get (for $m = 6, n = 3$),

$$3^6 - \binom{3}{1}(3-1)^6 + \binom{3}{2}(3-2)^6 = 540$$

Derangements I

Consider the hatcheck problem.

- An employee checks hats from $n$ customers.
- However, he forgets to tag them.
- When customer’s check-out their hats, they are given one at random.

What is the probability that no one will get their hat back?

Derangements II

This can be modeled using derangements: permutations of objects such that no element is in its original position.

The number of derangements of a set with $n$ elements is

$$D_n = n! \left[ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots (-1)^n \frac{1}{n!} \right]$$
Derangements III

Thus, the answer to the hatcheck problem is

$$\frac{D_n}{n!}$$

It's interesting to note that

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \cdots$$

So that the probability of the hatcheck problem converges;

$$\lim_{n \to \infty} \frac{D_n}{n!} = e^{-1} = .3679 \ldots$$

Derangements V

▶ In general:

$$(-1)^k \binom{n}{k} (n-k)! = \frac{n!}{k!}$$

▶ Last term will be when $$k = n$$ which is the identity permutation.

The Pigeonhole Principle I

The pigeonhole principle states that if there are more pigeons than there are roosts (pigeonholes), for at least one pigeonhole, at least two pigeons must be in it. This is a fundamental tool of elementary discrete mathematics. It is also known as the Dirichlet Drawer Principle.

Theorem (Pigeonhole Principle)

If $$k + 1$$ or more objects are placed into $$k$$ boxes, then there is at least one box containing two or more objects.

This is also known as the Dirichlet Drawer Principle.

Generalized Pigeonhole Principle I

Theorem

If $$N$$ objects are placed into $$k$$ boxes then there is at least one box containing at least

$$\left\lceil \frac{N}{k} \right\rceil$$

Example

In any group of 367 or more people, at least two of them must have been born on the same date.
Probabilistic Pigeonhole Principle I

A probabilistic generalization states that if \( n \) objects are randomly put into \( m \) boxes with uniform probability (each object is placed in a given box with probability \( 1/m \)) then at least one box will hold more than one object with probability,

\[
1 - \frac{m!}{(m-n)!m^n}
\]

Probabilistic Pigeonhole Principle II

Example

Among 10 people, what is the probability that two or more will have the same birthday?

Here, \( n = 10 \) and \( m = 365 \) (ignore leapyears). Thus, the probability that two will have the same birthday is

\[
1 - \frac{365!}{(365-10)!365^n} \approx .1169
\]

So less than a 12% probability!

Only 23 people required for a better than 50% (50.7%) probability

With only 57, we have a better than 99% probability

Probabilistic Pigeonhole Principle III

How many people do we need to have a better than 50% probability?

For what \( n \) is

\[
1 - \frac{365!}{(365-n)!365^n} \geq .50
\]

Surprisingly small: for \( n = 23 \), probability is greater than 50.7%!

Known as the “Birthday paradox”

Probabilistic Pigeonhole Principle IV

Derivation: consider the \( n \)-permutations of \( m \) pigeonholes:

\[
P(m, n) = \frac{m!}{(m-n)!} = m(m-1)(m-2) \cdots (m-n+1)
\]

These are the pigeonholes that we will evenly distribute \( n \) objects into. Order is important because the objects are distinct.

We place each object into distinct pigeonhole with probability \( 1/m \), so in total:

\[
\left( \frac{1}{m} \right)^n = \frac{1}{m^n}
\]

Thus the probability is:

\[
1 - \frac{m!}{(m-n)!m^n}
\]

Probabilistic Pigeonhole Principle V

Alternatively: consider choosing \( n \) bins (from a total of \( m \) bins to map \( n \) objects to:

\[
\binom{m}{n} = \frac{n!}{(m-n)!n!}
\]

But now consider actually mapping objects \( o_1, \ldots, o_n \) to each bin.

Here, order matters, but in addition, we only want to map one object to one bucket. That is, we want to count the total number of one-to-one functions from \( o_1, \ldots, o_n \) to the \( n \) bins we chose.

Thus:

\[
\frac{n!}{(m-n)!n!} \cdot n! = \frac{n!}{(m-n)!}
\]

Again, viewing allocation of objects to buckets, how many functions in total are there?

\[
m^n
\]

Probabilistic Pigeonhole Principle VI

Thus the probability of a random allocation resulting in all bins having 0 or 1 objects is

\[
\frac{n!}{(m-n)!} \div m^n = \frac{m!}{(m-n)!m^n}
\]
Example I

Show that in a room of \( n \) people with certain acquaintances, some pair must have the same number of acquaintances.

Note that this is equivalent to showing that any symmetric, irreflexive relation on \( n \) elements must have two elements with the same number of relations.

We’ll show by contradiction using the pigeonhole principle.

Assume to the contrary that every person has a different number of acquaintances; \( 0, 1, \ldots, n-1 \) (we cannot have \( n \) here because it is irreflexive). Are we done?

Example II

Show that in any list of ten nonnegative integers, \( a_0, \ldots, a_9 \), there is a string of consecutive items of the list \( a_i, a_{i+1}, \ldots, a_k \) whose sum is divisible by 10.

Consider the following 10 numbers.

\[
\begin{align*}
a_0 & = a_0 \\
a_0 + a_1 & = a_1 \\
a_0 + a_1 + a_2 & = a_2 \\
& \vdots \\
a_0 + a_1 + a_2 + \cdots + a_9 & = a_9
\end{align*}
\]

If any one of them is divisible by 10 then we are done.

Example III

Say 30 buses are to transport 2000 Cornhusker fans to Colorado. Each bus has 80 seats. Show that

1. One of the buses will have 14 empty seats.
2. One of the buses will carry at least 67 passengers.

For (1), the total number of seats is \( 30 \cdot 80 = 2400 \) seats. Thus there will be \( 2400 - 2000 = 400 \) empty seats total.

Example I

No, since we only have \( n \) people, this is okay (i.e. there are \( n \) possibilities).

We need to use the fact that acquaintanceship is a symmetric, irreflexive relation.

In particular, some person knows 0 people while another knows \( n - 1 \) people.

In other words, someone knows everyone, but there is also a person that knows no one.

Thus, we have reached a contradiction.

Example II

Otherwise, we observe that each of these numbers must be in one of the congruence classes

\[
1 \mod 10, 2 \mod 10, \ldots, 9 \mod 10
\]

By the pigeonhole principle, at least two of the integers above must lie in the same congruence class. Say \( a, a' \) lie in the congruence class \( k \mod 10 \).

Then

\[
(a - a') \equiv k - k \mod 10
\]

and so the difference \( (a - a') \) is divisible by 10.

Example III

By the generalized pigeonhole principle, with 400 empty seats among 30 buses, one bus will have at least

\[
\left\lceil \frac{400}{30} \right\rceil = 14
\]

empty seats.

For (2) above, by the pigeonhole principle, seating 2000 passengers among 30 buses, one will have at least

\[
\left\lceil \frac{2000}{30} \right\rceil = 67
\]

passengers.
Permutations I

A permutation of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of \( r \) elements of a set is called an \( r \)-permutation.

**Theorem**

The number of \( r \) permutations of a set with \( n \) distinct elements is

\[
P(n, r) = \frac{n!}{(n-r)!}
\]

In particular, \( P(n, n) = n! \)

Again, note here that order is important. It is necessary to distinguish in what cases order is important and in which it is not.

Permutations II

**Variation**

What if we allowed all 32 people to pair up in any combination?

- Number of permutations: \( 32! \)
- But a given pair, \( AB \) is the same as \( BA \)
- Correct for each pair: \( 2^{16} \)
- Now each pair’s ordering also doesn’t matter
- Correct for each such permutation: \( 16! \)

\[
\frac{32!}{2^{15}16!}
\]

Permutations I

**Variation**

Generalization: given \( kn \) objects, how many ways are there to form \( n \) groups of size \( k \)?

\[
\frac{(nk)!}{(k!)^n \cdot n!}
\]

Permutations

**Example I**

How many pairs of dance partners can be selected from a group of 12 women and 20 men?

The first woman can be partnered with any of the 20 men. The second with any of the remaining 19, etc.

To partner all 12 women, we have \( P(20, 12) \)

Another perspective: choose 12 men to include, then order them

Permutations

**Example II**

In how many ways can the English letters be arranged so that there are exactly ten letters between \( a \) and \( z \)?

The number of ways of arranging 10 letters between \( a \) and \( z \) is \( P(24, 10) \). Since we can choose either \( a \) or \( z \) to come first, there are \( 2P(24, 10) \) arrangements of this 12-letter block.

For the remaining 14 letters, there are \( P(15, 15) = 15! \) arrangements. In all, there are

\[
2P(24, 10) \cdot 15!
\]
Example

How many permutations of the letters a, b, c, d, e, f, g contain neither the pattern bge nor eaf?

The number of total permutations is \( P(7, 7) = 7! \).

If we fix the pattern bge, then we can consider it as a single block. Thus, the number of permutations with this pattern is \( P(5, 5) = 5! \).

Fixing the pattern eaf we have the same number, 5!.

Thus we have

\[
7! - 2(5!) + 3! = 4806
\]

Is this correct?

No. We have taken away too many permutations: ones containing both eaf and bge.

Here there are two cases, when eaf comes first and when bge comes first.

eaf cannot come before bge, so this is not a problem.

If bge comes first, it must be the case that we have bgeaf as a single block and so we have 3 blocks or 3! arrangements.

Altogether we have

\[
7! - 2(5!) + 3! = 4806
\]

A useful fact about combinations is that they are symmetric.

\[
\binom{n}{1} = \binom{n}{n - 1}
\]

\[
\binom{n}{2} = \binom{n}{n - 2}
\]

etc.
Combinations IV

Definition

This is formalized in the following corollary.

Corollary

Let \( n, k \) be nonnegative integers with \( k \leq n \), then

\[
\binom{n}{k} = \binom{n}{n - k}
\]

Combinations I

Example I

In the Powerball lottery, you pick five numbers between 1 and 59 and a single “powerball” number between 1 and 35. How many possible plays are there?

Order here doesn’t matter, so the number of ways of choosing five regular numbers is

\[
\binom{59}{5}
\]

Combinations II

Example I

We can choose among 35 power ball numbers. These events are not mutually exclusive, thus we use the product rule.

\[
35 \cdot \binom{59}{5} = 35 \cdot \frac{59!}{(59 - 5)!5!} = 175,223,510
\]

So the odds of winning are

\[
\frac{1}{175,223,510} < 5.70699 \times 10^{-9} = .00000000570699675\%
\]

Combinations III

Example II

Another perspective: what is the corresponding probability of 10 coin tosses ending up with 3 heads/7 tails?

It is the number of such outcomes divided by the total number of possible outcomes:

\[
\binom{10}{3}
\]

Why \( 2^{10} \)? Each toss was an independent event with 2 possible outcomes.

Another perspective: the total number of outcomes is equal to the total number of ways to choose: (0 tails, 10 heads), (1 tails, 9 heads), (2 tails, 8 heads), . . . (10 tails, 0 heads)

Which is:

\[
\sum_{i=0}^{10} \binom{10}{i} = 2^{10}
\]

Combinations II

Example II

However, this is the same as choosing 7 tails out of 10 coin tosses;

\[
\binom{10}{3} = \binom{10}{7} = 120
\]

This is a perfect illustration of the previous corollary.
Combinations IV
Example II

That is, the sum of binomial coefficients is equal to $2^n$

Gambler’s Fallacy

Say that we’ve flipped a coin and heads has appeared 9 times in a row.

What is the probability that the next flip will be tails? Heads?

Each event is independent: there is a fundamental difference between probabilities involving a sequence of events (parlays) and the probability of an event given a prior sequence.

August 18, 1913 Monte Carlo casino: black appeared 15 times in a row. Gamblers rushed to bet on red. Black would appear a total of 26 times in a row; many lost their bets thinking that red was “due”.

Krusty on why he bet against the Harlem Globetrotters: “I thought the Generals were due!”

Combinations I
Example III

Example

How many possible committees of five people can be chosen from 20 men and 12 women if

1. if exactly three men must be on each committee?
2. if at least four women must be on each committee?

Combinations II
Example III

For (1), we must choose 3 men from 20 then two women from 12. These are not mutually exclusive, thus the product rule applies.

\[
\binom{20}{3} \binom{12}{2}
\]

Combinations III
Example III

For (2), we consider two cases; the case where four women are chosen and the case where five women are chosen. These two cases are mutually exclusive so we use the addition rule.

For the first case we have

\[
\binom{20}{1} \binom{12}{4}
\]

Combinations IV
Example III

And for the second we have

\[
\binom{20}{0} \binom{12}{5}
\]

Together we have

\[
\binom{20}{1} \binom{12}{4} + \binom{20}{0} \binom{12}{5} = 10,692
\]
The number of r-combinations, \( \binom{n}{r} \), is also called a \textit{binomial coefficient}. They are the coefficients in the expansion of the expression (multivariate polynomial), \((x + y)^n\). A \textit{binomial} is a sum of two terms.

Expanding the summation, we have
\[
(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j
\]
For example,
\[
(x + y)^3 = (x + y)(x + y)(x + y)
= (x + y)(x^2 + 2xy + y^2)
= x^3 + 3x^2y + 3xy^2 + y^3
\]
By the Binomial Theorem, we have
\[
(3x + 4y)^n = \sum_{j=0}^{n} \binom{n}{j} (3x)^{n-j} (4y)^j
\]
So when \( j = 12 \), we have
\[
\binom{20}{12} (3x)^8 (4y)^{12}
\]
so the coefficient is \( \frac{20!}{12!8!} 3^8 4^{12} = 13866187326750720 \).

A lot of useful identities and facts come from the Binomial Theorem.

Corollary

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \quad n \geq 1
\]
\[
\sum_{k=0}^{n} x^k \binom{n}{k} = (1 + x)^n
\]
\[
\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n
\]
And many more.
Binomial Coefficients III

More

Most of these can be proven by either induction or by a combinatorial argument.

**Theorem (Vandermonde’s Identity)**

Let $m, n, r$ be nonnegative integers with $r$ not exceeding either $m$ or $n$. Then

\[
\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}
\]

**Corollary**

If $n$ is a nonnegative integer, then

\[
\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2
\]

**Corollary**

Let $n, r$ be nonnegative integers, $r \leq n$. Then

\[
\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}
\]

Binomial Coefficients I

Pascal’s Identity & Triangle

The following is known as Pascal’s Identity which gives a useful identity for efficiently computing binomial coefficients.

**Theorem (Pascal’s Identity)**

Let $n, k \in \mathbb{Z}^+$ with $n \geq k$. Then

\[
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}
\]

Pascal’s Identity forms the basis of a geometric object known as Pascal’s Triangle.

Pascal’s Triangle

\[
\begin{array}{cccccc}
& & & & & \\
& & & & 1 & \\
& & & 1 & 1 & \\
& & 1 & 2 & 1 & \\
& 1 & 3 & 3 & 1 & \\
1 & 4 & 6 & 4 & 1 & \\
& 1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\]

Pascal’s Triangle

\[
\begin{array}{cccccc}
& & & & & \\
& & & & 1 & \\
& & & 1 & 1 & \\
& & 1 & 2 & 1 & \\
& 1 & 3 & 3 & 1 & \\
1 & 4 & 6 & 4 & 1 & \\
& 1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\]

Pascal’s Triangle

\[
\begin{array}{cccccc}
& & & & & \\
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& & 1 & 2 & 1 & \\
& 1 & 3 & 3 & 1 & \\
1 & 4 & 6 & 4 & 1 & \\
& 1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\]

Pascal’s Triangle

\[
\begin{array}{cccccc}
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& & & & 1 & \\
& & & 1 & 1 & \\
& & 1 & 2 & 1 & \\
& 1 & 3 & 3 & 1 & \\
1 & 4 & 6 & 4 & 1 & \\
& 1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\]
Generalized Permutations I

Sometimes we are concerned with permutations and combinations in which repetitions are allowed.

**Theorem**

The number of \( r \)-permutations of a set of \( n \) objects with repetition allowed is \( n^r \).

Generalized Combinations I

Theorem

There are \( \binom{n+r-1}{r} = \binom{n+r-1}{n-1} \) \( r \)-combinations from a set with \( n \) elements when repetition of elements is allowed.

Generalized Combinations II

To see this consider:
- \( n-1 \) bars (so \( n \) “cells” that represent that types of items)
- \( r \) stars: (if \( x \) stars are placed into a given cell, its the number of elements of that type that we choose)
- So \( n-1 + r \) “things” to be arranged
- However, we’ve over counted: stars and bars are indistinguishable
- Any sequence of contiguous stars/bars is the same under any ordering, so we need to divide out by permutations of \( n-1 \) and \( r \):

\[
\frac{(n-1+r)!}{(n-1)!r!} = \frac{(n-1+r)!}{(n+r-1-r)!r!} = \binom{n+r-1}{r}
\]

Generalized Combinations III

Example

“Mississippi” contains 4 distinct letters, \( M, i, s \) and \( p \); with 1, 4, 4, 2 occurrences respectively.

Therefore there are \( \frac{11!}{1!4!4!2!} \) permutations.
**Distinguishable Objects into Distinguishable Boxes**

Example: how many ways are there to deal a 52 card deck into 5 card hands to four players?

\[
\binom{52}{5} \binom{47}{5} \binom{42}{5} \binom{37}{5}
\]

**Theorem**

The number of ways to distribute \( n \) distinguishable objects into \( k \) distinguishable boxes such that each box has \( n_i \) objects for \( i = 1, \ldots, k \) is

\[
\frac{n!}{n_1!n_2! \cdots n_k!}
\]

Card example: the 5 box should contain the remaining 32 undealt cards

---

**Distinguishable Objects into Indistinguishable Boxes**

Distinguishable objects into indistinguishable boxes: not the same as vice versa.

Equivalent to Stirling numbers of the 2nd kind (the \( \frac{1}{k!} \) factor recognizes that boxes are indistinguishable and corrects for it)

**Theorem**

The number of ways to distribute \( n \) distinguishable objects into \( k \) indistinguishable boxes is

\[
\frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n
\]

---

**Generating Function Example I**

How many ways are there to make change for a dollar (using half-dollars, quarters, dimes, nickels, pennies)?

The generating function:

\[
C(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})}
\]

gives us the answer: for any amount of change \( c \), we want the coefficient of \( x^c \) in the expansion of \( C(x) \).

For \( c = 100 \), the coefficient of \( x^{100} \) is 292.

---

**Indistinguishable Objects into Distinguishable Boxes**

Indistinguishable objects into distinguishable boxes: equivalent to \( \binom{n}{k} \) combinations of \( k \) elements when repetition is allowed (parameters are switched).

**Theorem**

The number of ways to distribute \( n + k - 1 \) distinguishable objects into \( k \) distinguishable boxes is

\[
\binom{k + n - 1}{n} = \binom{k + n - 1}{k - 1}
\]

---

**Related: Set Partitions I**

The number of ways to partition a set into disjoint non-empty subsets (such that their union is the original set).

Example: \( S = \{1, 2, 3\} \) then the partitions are:

\[
\{\{a\}, \{b\}, \{c\}\}, \{\{a\}, \{b, c\}\}, \{\{b\}, \{a, c\}\}, \{\{c\}, \{a, b\}\}, \{\{a, b, c\}\}
\]

**Theorem**

- Far more complicated
- Example: 6 copies of the same book into 4 identical boxes: \( (6), (5, 1), (4, 2), (4, 1, 1), (3, 3), \ldots \)
- Equivalent to partitions of an integer sum
- No closed form, requires a generating function:
Related: Set Partitions II

The number of ways to partition a set of size \( n \) into disjoint non-empty subsets corresponds to the \( n \)-th Bell Number:

\[
B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k
\]

Note: \( B_0 = 1, B_1 = 1, B_2 = 2, B_3 = 5, B_4 = 15, 52, 203, 877, 4140, 21147, 115975, \ldots \)

Basic Idea:

- Consider a new element \( e_{n+1} \)
- It could be in its own partition, thus there are \( \binom{n}{0} \cdot B_n \) partitionings of the remaining elements

Related: Set Partitions III

- It could be paired with one other element, leaving \( \binom{n}{1} \cdot B_{n-1} \) partitionings of remaining elements
- It could be paired with two other element, leaving \( \binom{n}{2} \cdot B_{n-2} \) partitionings of remaining elements
- All the way to \( \binom{n}{n} \cdot B_0 \)

Each Bell number is the sum of Stirling numbers of the second kind:

\[
B_n = \sum_{k=0}^{n} \{ n \}_k
\]

Idea: Stirling numbers give the number of ways to partition into \( k \) non-empty subsets; summing over all such \( k \) gives us the formula.

Generating Permutations & Combinations I

Introduction

In general, it is inefficient to solve a problem by considering all permutations or combinations since there are an exponential number of such arrangements.

Nevertheless, for many problems, no better approach is known. When exact solutions are needed, back-tracking algorithms are used.

Generating permutations or combinations are sometimes the basis of these algorithms.

Generating Permutations & Combinations II

Introduction

Example (Traveling Sales Person Problem)

Consider a salesman that must visit \( n \) different cities. He wishes to visit them in an order such that his overall distance traveled is minimized.

The only known way of solving this problem exactly is to try all \( n! \) possible routes.

Generating Permutations & Combinations III

Introduction

This problem is one of hundreds of NP-complete problems for which no known efficient algorithms exist. Indeed, it is believed that no efficient algorithms exist. (Actually, Euclidean TSP is not even known to be in NP!)

The only known way of solving this problem exactly is to try all \( n! \) possible routes.

We give several algorithms for generating these combinatorial objects.

Generating Combinations I

Recall that combinations are simply all possible subsets of size \( r \). For our purposes, we will consider generating subsets of

\[
\{1, 2, 3, \ldots, n\}
\]

The algorithm works as follows.

- Start with \( \{1, \ldots, r\} \)
- Assume that we have \( a_1 a_2 \cdots a_r \), we want the next combination.
- Locate the last element \( a_i \) such that \( a_i \neq n - r + i \).
- Replace \( a_i \) with \( a_i + 1 \).
- Replace \( a_j \) with \( a_i + j - i \) for \( j = i+1, i+2, \ldots, r \).
Generating Combinations II

The following is pseudocode for this procedure.

Algorithm (Next r-Combination)

\[
\begin{array}{l}
\textbf{Input} : \text{A set of } n \text{ elements and an r-combination, } a_1 \cdots a_r. \\
\textbf{Output} : \text{The next r-combination.}
\end{array}
\]

1. \( i = r \)
2. WHILE \( a_i = n - r + i \) DO
   3. \( i = i - 1 \)
4. END
5. \( a_i = a_i + 1 \)
6. FOR \( j = (i+1) \cdots r \) DO
   7. \( a_j = a_i + j - i \)
8. END

Generating Permutations

The text gives an algorithm to generate permutations in lexicographic order. Essentially the algorithm works as follows.

Given a permutation,
- Choose the left-most pair \( a_j, a_{j+1} \) where \( a_j < a_{j+1} \).
- Choose the least item to the right of \( a_j \) greater than \( a_j \).
- Swap this item and \( a_j \).
- Arrange the remaining (to the right) items in order.

Generating Permutations II

Often there is no reason to generate permutations in lexicographic order. Moreover, even though generating permutations is inefficient in itself, lexicographic order induces even more work.

An alternate method is to fix an element, then recursively permute the \( n-1 \) remaining elements.

Another method has the following attractive properties.
- It is bottom-up (non-recursive).
- It induces a minimal-change between each permutation.

Generating Permutations III

Example

Find the next 3-combination of the set \{1, 2, 3, 4, 5\} after \{1, 4, 5\}.

Here, \( a_1 = 7 \), \( a_2 = 4 \), \( a_3 = 5 \).

Thus, we set
\[
\begin{align*}
   a_1 &= a_1 + 1 = 2 \\
   a_2 &= a_1 + 2 - 1 = 3 \\
   a_3 &= a_1 + 3 - 1 = 4 \\
\end{align*}
\]

So the next r-combination is \{2, 3, 4\}.

Generating Permutations

Lexicographic Order

Algorithm (Next Permutation (Lexicographic Order))

\[
\begin{array}{l}
\textbf{Input} : \text{A set of } n \text{ elements and an r-permutation, } a_1 \cdots a_r. \\
\textbf{Output} : \text{The next r-permutation.}
\end{array}
\]

1. \( j = n - 1 \)
2. WHILE \( a_j > a_{j+1} \) DO
   3. \( j = j - 1 \)
4. END
5. // \( j \) is the largest subscript with \( a_j < a_{j+1} \)
6. \( k = n \)
7. WHILE \( a_j > a_k \) DO
   8. \( k = k - 1 \)
9. END
10. // \( a_k \) is the smallest integer greater than \( a_j \) to the right of \( a_j \)
11. \( \text{swap}(a_j, a_k) \)
12. \( r = n \)
13. \( s = j + 1 \)
14. WHILE \( r > s \) DO
15. \( \text{swap}(a_r, a_s) \)
16. \( r = r - 1 \)
17. \( s = s + 1 \)

The algorithm is known as the Johnson-Trotter algorithm.

We associate a direction to each element, for example:
\[
\begin{array}{cccc}
3 & 2 & 1 & 4 \\
\end{array}
\]

A component is mobile if its direction points to an adjacent component that is smaller than itself. Here 3 and 4 are mobile and 1 and 2 are not.
Generating Permutations III

Algorithm (JohnsonTrotter)

Input: An integer \(n\).
Output: All possible permutations of \((1, 2, \ldots, n)\).

1. \(\pi \leftarrow \underbrace{1}_{\text{1}} \underbrace{2}_{\text{2}} \ldots \underbrace{n}_{\text{n}}\)
2. \(\text{while There exists a mobile integer } k \in \pi \text{ do}\)
3. \(k \leftarrow \text{largest mobile integer}\)
4. \(\text{swap } k \text{ and the adjacent integer } k \text{ points to}\)
5. \(\text{reverse direction of all integers } > k\)
6. \(\text{Output } \pi\)
7. \(\text{end}\)

Johnson-Trotter Examples

Example A: consider permutations of \((1, \ldots, 6)\):

\[
\begin{align*}
1 & \quad 3 & \quad 5 & \quad 6 & \quad 4 & \quad 2 \\
2, & & & & & \quad \text{is mobile, so swap } 6, 5; \quad \text{no orientation changes:}
\end{align*}
\]

\[
\begin{align*}
1 & \quad 3 & \quad 5 & \quad 6 & \quad 4 & \quad 2 \\
2, & & & & & \quad \text{is the only mobile element, flip } 1, 2; \quad \text{orientation of } 4, 3, 5, 6 \text{ gets reversed:}
\end{align*}
\]

More Examples

As always, the best way to learn new concepts is through practice and examples.

Example II

Let \(S, T\) be sets such that \(|S| = n, |T| = m\). How many functions are there mapping \(f : S \rightarrow T\)? How many of these functions are one-to-one?

A function simply maps each \(s_i\) to some \(t_j\), thus for each \(n\) we can choose to send it to any of the elements in \(T\).
Example: Counting Primes I

Example
Give an estimate for how many 70 bit primes there are.

Recall that the number of primes not more than \( n \) is about

\[
\frac{n}{\ln n}
\]

Using this fact, the number of primes not exceeding \( 2^{70} \) is

\[
\frac{2^{70}}{\ln 2^{70}}
\]

Thus the difference is

\[
\frac{2^{70}}{\ln 2^{70}} - \frac{2^{69}}{\ln 2^{69}} \approx 1.19896 \times 10^{19}
\]

Example: Counting Primes II

However, we have over counted—we’ve counted 69-bit, 68-bit, etc primes as well.

Example: Counting Functions I III

In particular, for the second element, \( s_2 \), we now have \( m - 1 \) choices. Proceeding in this manner, \( s_3 \) will have \( m - 2 \) choices, etc. Thus we have

\[
m \cdot (m - 1) \cdot (m - 2) \cdots \cdot (m - (n - 2)) \cdot (m - (n - 1))
\]

An alternative way of thinking about this problem is by using the choose operator: we need to choose \( n \) elements from a set of size \( m \) for our mapping:

\[
\binom{m}{n} = \frac{m!}{(m - n)!n!}
\]

Example: Counting Functions II

Recall this question from the 1st exam:

Example
Let \( S = \{1, 2, 3\} \), \( T = \{a, b\} \). How many onto functions are there mapping \( S \to T \)? How many one-to-one functions are there mapping \( T \to S \)?

Example: Counting Functions I IV

Once we have chosen this set, we now consider all permutations of the mapping, i.e. \( n! \) different mappings for this set. Thus, the number of such mappings is

\[
\frac{m!}{(m - n)!n!} \cdot n! = \frac{m!}{(m - n)!n!} \cdot n!
\]

Example: Counting Functions I II

Each of these is an independent event, so we apply the multiplication rule:

\[
m \times m \times \cdots m = m^n
\]

\( n \) times

If we wish \( f \) to be one-to-one, we must have that \( n \leq m \), otherwise we can easily answer 0.

Now, each \( s_i \) must be mapped to a unique element in \( T \). For \( s_1 \), we have \( m \) choices. However, once we have made a mapping (say \( t_j \)), we cannot map subsequent elements to \( t_j \) again.
Example: More sets I

Example

How many integers in the range $1 \leq k \leq 100$ are divisible by 2 or 3?

Let

\[ A = \{ x | 1 \leq x \leq 100, 2 \mid x \} \]
\[ B = \{ y | 1 \leq x \leq 100, 3 \mid y \} \]

Clearly, $|A| = 50, |B| = \lfloor \frac{100}{3} \rfloor = 33$, so is it true that $|A \cup B| = 50 + 33 = 83$?

Example: More sets II

No; we've over counted again—any integer divisible by 6 will be in both sets. How much did we over count?

The number of integers between 1 and 100 divisible by 6 is $\lfloor \frac{100}{6} \rfloor = 16$, so the answer to the original question is

$|A \cup B| = (50 + 33) - 16 = 67$