Asymptotics
Computer Science & Engineering 235: Discrete Mathematics

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Introduction I

Recall that we are really only interested in the Order of Growth of an algorithm’s complexity.

How well does the algorithm perform as the input size grows; 

\[ n \to \infty \]

We’ve seen how to mathematically evaluate the cost functions of algorithms with respect to their input size \( n \) and their elementary operation.

However, it suffices to simply measure a cost function’s asymptotic behavior.

In practice, specific hardware, implementation, languages, etc. will greatly affect how the algorithm behaves. However, we want to study and analyze algorithms in and of themselves, independent of such factors.

For example, an algorithm that executes its elementary operation \( 10n \) times is better than one which executes it \( .005n^2 \) times.

Moreover, algorithms that have running times \( n^2 \) and \( 2000n^2 \) are considered to be asymptotically equivalent.

Big-O Definition

**Definition**

Let \( f \) and \( g \) be two functions \( f, g : \mathbb{N} \to \mathbb{R}^+ \). We say that

\[ f(n) \in O(g(n)) \]

(read: \( f \) is Big-“O” of \( g \)) if there exists a constant \( c \in \mathbb{R}^+ \) and \( n_0 \in \mathbb{N} \) such that for every integer \( n \geq n_0 \),

\[ f(n) \leq cg(n) \]

▶ Big-O is actually Omicron, but it suffices to write “O”
▶ Intuition: \( f \) is (asymptotically) less than or equal to \( g \)
▶ Big-O gives an asymptotic upper bound

Big-Omega Definition

**Definition**

Let \( f \) and \( g \) be two functions \( f, g : \mathbb{N} \to \mathbb{R}^+ \). We say that

\[ f(n) \in \Omega(g(n)) \]

(read: \( f \) is Big-Omega of \( g \)) if there exist \( c \in \mathbb{R}^+ \) and \( n_0 \in \mathbb{N} \) such that for every integer \( n \geq n_0 \),

\[ f(n) \geq cg(n) \]

▶ Intuition: \( f \) is (asymptotically) greater than or equal to \( g \)
▶ Big-Omega gives an asymptotic lower bound
Definitional Proof - Example I

Definitional Proof

Big-Theta Definition

Definition
Let \( f \) and \( g \) be two functions \( f, g : \mathbb{N} \to \mathbb{R}^+ \). We say that
\[
f(n) \in \Theta(g(n))
\]
(read: \( f \) is asymptotically equal to \( g \)).

▶ \( f \) is bounded above and below by \( g \).
▶ Big-Theta gives an asymptotic equivalence.

Illustrative Example

▶ \( f(n) \in O(g(n)) \) if \( g(n) \) eventually dominates \( f(n) \)
▶ Let \( f(n) = 100n^2, g(n) = n^3 \)
▶ For \( n < 100 \), \( f(n) > g(n) \)
▶ At \( n = 100 \), \( f(n) = g(n) \)
▶ For \( n > 100 \), \( g(n) > f(n) \)

Figure: \( g(n) \) dominates for all \( n > 100 \)

Asymptotic Properties I

Theorem

For \( f_1(n) \in O(g_1(n)) \) and \( f_2 \in O(g_2(n)) \),
\[
f_1(n) + f_2(n) \in O(\max\{g_1(n), g_2(n)\})
\]

This property implies that we can ignore lower order terms. In particular, for any polynomial \( p(n) \) with degree \( k \), \( p(n) \in O(n^k) \).

In addition, this gives us justification for ignoring constant coefficients. That is, for any function \( f(n) \) and positive constant \( c \),
\[
 cf(n) \in \Theta(f(n))
\]

Corollary

For positive functions, \( f(n) \) and \( g(n) \) the following hold:

▶ \( f(n) \in \Theta(g(n)) \iff f(n) \in O(g(n)) \text{ and } f(n) \in \Omega(g(n)) \)

The proof is left as an exercise.

Asymptotic Properties II

Some obvious properties also follow from the definition.

Example

Let \( f(n) = 21n^2 + n \) and \( g(n) = n^3 \). Our intuition should tell us that \( f(n) \in O(g(n)) \). Simply using the definition confirms this:
\[
21n^2 + n \leq cn^3
\]
holds for, say \( c = 3 \) and for all \( n \geq n_0 = 8 \) (in fact, an infinite number of pairs can satisfy this equation).

Proving an asymptotic relationship between two given functions \( f(n) \) and \( g(n) \) can be done intuitively for most of the functions you will encounter; all polynomials for example. However, this does not suffice as a formal proof.

To prove a relationship of the form \( f(n) \in \Delta(g(n)) \) where \( \Delta \) is one of \( O, \Omega, \Theta \), can be done simply using the definitions.
Asymptotic Proof Techniques
Definitional Proof - Example II

Example
Let \( f(n) = n^2 + n \) and \( g(n) = n^3 \). Find a tight bound of the form \( f(n) \in \Delta(g(n)) \).

Our intuition tells us that
\[
f(n) \in \Theta(n^3)
\]

Asymptotic Proof Techniques
Definitional Proof - Example III

Example
Let \( f(n) = n^3 + 4n^2 \) and \( g(n) = n^2 \). Find a tight bound of the form \( f(n) \in \Delta(g(n)) \).

Here, our intuition should tell us that
\[
f(n) \in \Omega(g(n))
\]

Limit Method

Now try this one:
\[
f(n) = n^{50} + 12n^3 \log^4 n - 1243n^{12} + 245n^6 \log n + 121 \log^3 n - 1243n^{12} + 245n^6 \log n + 12\]
\[
g(n) = 12n^{50} + 245n^6 \log n + 12 \log^3 n - \log n
\]

Using the formal definitions can be very tedious especially when one has very complex functions. It is much better to use the Limit Method which uses concepts from calculus.

Limit Method Process

Say we have functions \( f(n) \) and \( g(n) \). We set up a limit quotient between \( f \) and \( g \) as follows:
\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \begin{cases} 
0 & \text{then } f(n) \in \Theta(g(n)) \\
c > 0 \text{ then } f(n) \in \Theta(g(n)) \\
\infty & \text{then } f(n) \in \Omega(g(n))
\end{cases}
\]

- Justifications for the above can be proven using calculus, but for our purposes the limit method will be sufficient for showing asymptotic inclusions.
- Always try to look for algebraic simplifications first.
l'Hôpital's Rule

Example: prove that \( \log n \in O(n) \); setting up a limit:

\[
\lim_{n \to \infty} \frac{\log n}{n} = ?
\]

If \( f \) and \( g \) both diverge or converge on zero, then you need to apply l'Hôpital's Rule.

Theorem

(l'Hôpital's Rule) Let \( f \) and \( g \), if the limit between the quotient \( \frac{f(n)}{g(n)} \) exists, it is equal to the limit of the derivative of the denominator and the numerator.

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{f'(n)}{g'(n)}
\]

l'Hôpital's Rule

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\]

If \( f \) and \( g \) both diverge or converge on zero, then you need to apply l'Hôpital's Rule.

Justification

Why do we have to use l'Hôpital's Rule? Consider the following function:

\[ f(x) = \frac{\sin x}{x} \]

What is \( f(0) \)?

Clearly, \( \sin 0 = 0 \) so you may say that \( f(0) = 0 \). However, the denominator is also zero so you may say \( f(0) = \infty \), but both are wrong.

l'Hôpital's Rule II

Justification

Observe the graph of \( f(x) \):

\[ f(x) = \frac{\sin x}{x} \]

Figure: \( f(x) = \sin x \)

l'Hôpital's Rule III

Justification

Clearly, though \( f(x) \) is undefined at \( x = 0 \), the limit still exists.

Applying l'Hôpital’s Rule gives us the correct answer:

\[
\lim_{x \to 0} \frac{\sin x'}{x'} = \frac{\cos x}{1} = 1
\]

Limit Method

Example 1

Example

Let \( f(n) = 2^n \), \( g(n) = 3^n \). Determine a tight inclusion of the form \( f(n) \in \Delta(g(n)) \).

What’s our intuition in this case?

Limit Method

Example 1 - Proof A

Proof.

▶ We prove using limits.
▶ We set up our limit,

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \frac{2^n}{3^n}
\]

▶ Using l'Hôpital's Rule will get you no where.

\[
\frac{2^n}{3^n} = \left(\frac{\ln 2}{\ln 3}\right)^n
\]

Both numerator and denominator still diverge. We’ll have to use an algebraic simplification.

Limit Method

Example 1 - Proof A

Proof.

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\[
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\]

Both numerator and denominator still diverge. We’ll have to use an algebraic simplification.
Limit Method
Example 1 - Proof A

Proof.
▶ We prove using limits.
▶ We set up our limit,
\[ \lim_{n \to \infty} \frac{f(n)}{g(n)} = \log_2 n \log_3 n^2 \]
▶ Here, we have to use the change of base formula for logarithms:
\[ \log_\alpha n = \log_c n \log_\alpha c \]
▶ Therefore we conclude that the quotient converges to zero thus,
\[ 2^n \in O(3^n) \]

Limit Method
Example 2 - Proof B

Continued.
▶ And we get that
\[ \lim_{n \to \infty} \frac{f(n)}{g(n)} = \log_2 n \log_3 n^2 \]
= \log_2 n \frac{2 \log_2 n}{\log_2 3} \approx 0.7924 \ldots
▶ So we conclude that \( f(n) \in \Theta(g(n)). \)

Limit Method
Example 2

Example
Let \( f(n) = \log_2 n, \ g(n) = \log_3 n^2. \) Determine a tight inclusion of the form \( f(n) \in \Delta(g(n)). \)
What's our intuition in this case?

Limit Properties

A useful property of limits is that the composition of functions is preserved.

Lemma

For the composition \( \circ \) of addition, subtraction, multiplication and division, if the limits exist (that is, they converge), then
\[ \lim_{n \to \infty} f_2(n) \circ \lim_{n \to \infty} f_2(n) = \lim_{n \to \infty} f_1(n) \circ f_2(n) \]

Useful Identities & Derivatives

Some useful derivatives that you should memorize include
▶ \( (nk)' = kn^{k-1} \)
▶ \( (\log_\alpha n)' = \frac{1}{\ln(\alpha) n} \)
▶ \( (f_1(n)f_2(n))' = f_1'(n)f_2(n) + f_1(n)f_2'(n) \) (product rule)
▶ \( (e^n)' = \ln(e)e^n \leftarrow \text{Careful!} \)

Log Identities
▶ Change of Base Formula: \( \log_\alpha n = \frac{\log_b n}{\log_b \alpha} \)
▶ \( \log(n^k) = k \log(n) \)
▶ \( \log(ab) = \log(a) + \log(b) \)
Efficiency Classes

<table>
<thead>
<tr>
<th>Category</th>
<th>Asymptotic Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>Logarithmic</td>
<td>$O(\log(n))$</td>
</tr>
<tr>
<td>Linear</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>Polylogarithmic</td>
<td>$O(\log^k(n))$</td>
</tr>
<tr>
<td>Quadratic</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td>Cubic</td>
<td>$O(n^3)$</td>
</tr>
<tr>
<td>Polynomial</td>
<td>$O(n^k)$ for any $k &gt; 0$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$O(2^n)$</td>
</tr>
<tr>
<td>Super-Exponential</td>
<td>$O(2^{f(n)})$ for $f(n) \in \Omega(n)$ for example, $n!$</td>
</tr>
</tbody>
</table>

Table: Some Efficiency Classes

Little-o Definition I

Asymptotic notation provides loose bounds (except for $\Theta$). These bounds are sufficient for algorithm analysis but for many applications it is much more accurate to use little-o and little-omega notation.

Conveniently, the definition of $o$ and $\omega$ are exactly what we get when we use the limit method.

Little-o Definition II

Definition
Let $f$ and $g$ be two functions, $f, g : \mathbb{N} \to \mathbb{R}^+$. We say that $f(n) \in o(g(n))$ (read: $f$ is in little-o of $g$) if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

Alternatively, $f(n) \in o(g(n))$ means that for any real number $c > 0$, there exists $n_0$ such that $f(n) < cg(n)$ for all $n \geq n_0$.

Little-o can be interpreted as “$f$ is strictly less than (never equal to) $g$.”

Little-omega Definition

Definition
Let $f$ and $g$ be two functions, $f, g : \mathbb{N} \to \mathbb{R}^+$. We say that $f(n) \in \omega(g(n))$ (read: $f$ is little-omega of $g$) if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$$

Alternatively, $f(n) \in \omega(g(n))$ means that for any real number $c > 0$, there exists $n_0$ such that $f(n) > cg(n)$ for all $n \geq n_0$.

Little-omega can be interpreted as “$f$ is strictly greater than (never equal to) $g$.”

Little-o Examples

Some examples that we provide without proof:
- $\sqrt{n} \in o(n)$
- $n \in o(n \log \log n)$
- $n \log \log n \in o(n \log n)$
- $n \log n \in o(n^2)$
- $n^\alpha \in o(n^\beta)$ for any $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$
- $\alpha n \in o(\beta n)$ for any $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$
- $n \in o(n^{1+\epsilon})$
- $f(n) \notin o(f(n))$

Little Asymptotics I

Properties
Note that though simply using the limit method gives us these bounds. That is,

$$f(n) \in o(g(n)) \Rightarrow f(n) \in O(g(n))$$

and

$$f(n) \in \omega(g(n)) \Rightarrow f(n) \in \Omega(g(n))$$

However, the converses do not hold.
Little Asymptotics II
Properties

Here are some examples of little-omega relationships.

- $n! \in \omega(2^n)$
- $n^n \in \omega(n^{n^\epsilon})$
- $n^n \in o(n^{n^{\epsilon+\epsilon}})$
- $f(n) \not\in \omega(f(n))$

Note that $f(n) \in o(g(n)) \iff g(n) \in \omega(f(n))$ still holds.
Also, $f(n) \not\in o(g(n))$ and $f(n) \not\in \omega(g(n)) \iff f(n) \in \Theta(g(n))$.

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Soft-O Notation I

Logarithmic factors contribute very little as $n \to \infty$.
Soft-O notation, $\tilde{O}$ is used to ignore log factors:
If $f(n) \in O(g(n) \cdot \log^{k} n)$ then $f(n) \in \tilde{O}(g(n))$.

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Summary

Asymptotics is easy, but remember:
- Always look for algebraic simplifications
- You must always give a rigorous proof
- Using the limit method is always the best
- Always show l'Hôpital’s Rule if need be
- Give as simple (and tight) expressions as possible