Instructions Follow all instructions carefully, failure to do so may result in points being deducted. Clearly label each problem and submit the answers in order. Print out a copy of this cover sheet and staple it to the front of your assignment for grading. Be sure to show sufficient work to justify your answers. When asked to prove something, you must give a formal, rigorous, and complete proof. The CSE academic dishonesty policy is in effect (see https://cse.unl.edu/academic-integrity). You are highly encouraged to typeset your assignment using \LaTeX; if your answers are not legible, you may be required to use \LaTeX in future assignments.

Partner Policy You have the option of working in pairs for this assignment. You always have the option of working alone if you chose. If you choose to work as a pair, you must adhere to these guidelines:

1. You must work on all problems together. You may not simply partition the work between you.
2. You should not discuss problems with other groups or individuals beyond general questions.
3. Hand in only one hard copy with both of your names and logins. The first author should hand in any soft copies of programs.

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1. **4 points** For each of the following sets, 1) determine whether there is a one-to-one function \( f : S \rightarrow T \); 2) determine whether there is an onto function \( f : S \rightarrow T \); and 3) determine if there is a bijective function \( f : S \rightarrow T \). For each (1–3) if such a function exists, explicitly give it. If no function exists give a short explanation of why.

(a) \( S = \{1, 2, 3\}, T = \{a, b, c\} \)
(b) \( S = \{a, b\}, T = \{1, 2, 3, 4\} \)
(c) \( S = \{1, 2, 3, 4\}, T = \{a, b\} \)

2. **6 points** Let \( S = \{1, 2\} \) and \( T = \{a, b, c\} \).

(a) How many unique functions are there mapping \( S \rightarrow T \)?
(b) How many unique functions are there mapping \( T \rightarrow S \)?
(c) How many onto (surjective) functions are there mapping \( S \rightarrow T \)?
(d) How many onto (surjective) functions are there mapping \( T \rightarrow S \) (hint: think of how many non onto functions there are)?
(e) How many one-to-one (injective) functions are there mapping \( S \rightarrow T \)?
(f) How many one-to-one (injective) functions are there mapping \( T \rightarrow S \)?
(g) Let \( f : S \rightarrow T \), is it possible to define \( f^{-1} \)? Why or why not?

3. **5 points** Determine whether each of the functions below is onto, and/or one-to-one, prove your answers.

(a) \( f : \mathbb{Z} \rightarrow \mathbb{Z}, f(x) = x - 1 \)
(b) \( f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = -3x^2 - 7 \)
(c) \( f : \mathbb{Z} \rightarrow \mathbb{Z}, f(x) = [x] \)
(d) \( f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x \)
(e) \( f : \mathbb{R} \rightarrow \mathbb{Z}, f(x) = [x] \)

4. **6 points** Determine whether each of the functions below is onto, and/or one-to-one for \( f : \mathbb{Z} \rightarrow \mathbb{Z} \), prove your answers.

(a) \( f(x) = 5x - 3 \)
(b) \( f(x) = 2x^3 \)
(c) \( f(x) = (2x - 2)^2 \)
(d) \( f(x) = \sqrt{x} \)

5. **6 points** Determine whether each of the functions below is onto, and/or one-to-one for \( f : \mathbb{R} \rightarrow \mathbb{R} \), prove your answers.

(a) \( f(x) = 5x - 3 \)
(b) \( f(x) = 2x^3 \)
(c) \( f(x) = (2x - 2)^2 \)
(d) \( f(x) = \sqrt{x} \)

6. **4 points** Define the following functions (assume that the domains/codomains are defined such that each composition is valid): \( f(x) = 2x, g(x) = \frac{x}{1+x}, h(x) = \sqrt{x} \). Find

(a) \( f \circ g \circ h \)
(b) \( h \circ g \circ f \)
7. 4 points  Find inverses of the following functions (assume that the domains/codomains are defined such that each function is a bijection).
   
   (a) \( f(x) = 5x - 3 \)
   (b) \( f(x) = 2x^3 \)
   (c) \( f(x) = (2x - 2)^2 \)
   (d) \( f(x) = \frac{\sqrt{x}}{6} \)

8. 4 points  Let \( f(x) \) and \( g(x) \) be two linear functions (a function is \textit{linear} if \( f(x) = ax + b \) for \( a, b \in \mathbb{R} \)).
   
   a. Prove or disprove: \( f \circ g = g \circ f \).
   
   b. Prove or disprove: \( f \circ g \) and \( g \circ f \) are linear functions.

9. 8 points  Let \( R \) be a relation on a set \( A = \{a_1, \ldots, a_n\} \) of size \( n \). Let \( M_R \) be the 0-1 matrix representing \( R \) (i.e., the entry \( m_{ij} = 1 \) if \( (a_i, a_j) \in R \) and zero otherwise).
   
   (a) How many unique relations are there on \( A \) (in terms of \( n \))? 
   
   (b) The \textit{complement} relation is defined as 
   
   \[ \overline{R} = \{(a, b) \mid (a, b) \not\in R\} \] 
   
   Say that the number of nonzero entries in \( M_R \) (that is, the number of 1s) is \( k \). How many nonzero entries are there in \( M_{\overline{R}} \)? Briefly justify your answer.
   
   (c) How many reflexive relations are there on a set of size \( n \)? Briefly justify your answer.
   
   (d) How many symmetric relations are there on a set of size \( n \)? Briefly justify your answer.

10. 6 points  Define the following relation on the set of integers. 
   
   \( R_1 = \{(a, b) \mid a > b, \ a, b \in \mathbb{Z}\} \)
   
   (a) Give an element that is in \( R_1 \)
   
   (b) Give an element that is not in \( R_1 \).
   
   (c) Give an element that is in \( R_1 \circ R_1 \)
   
   (d) Give an element that is not in \( R_1 \circ R_1 \).
   
   (e) Give an element that is in \( R_1^3 \)
   
   (f) Give an element that is not in \( R_1^3 \)
   
   (g) Give a general (set builder) definition for \( R_1^n \) in terms of \( a, b \) and \( n \).

11. 10 points  Prove the following. A relation \( R \) is asymmetric if and only if \( R \) is irreflexive and antisymmetric. Note: a relation \( R \) on the set \( A \) is \textit{irreflexive} if for every \( a \in A \), \( (a, a) \not\in R \). That is, \( R \) is irreflexive if no element in \( A \) is related to itself.
   
   Hint: write the definition of what it means to be asymmetric, then “add” the contradiction:
   
   \( (a = b) \land (a \neq b) \)

12. 2 points  Prove or disprove: if \( R \) is antisymmetric, then \( R \) is asymmetric
13. **60 points Programming Assignment** – In this programming assignment, you will implement a Java class, `Relation.java` that will model relations on a set. The skeleton class is available for download on the webpage. You must correctly implement all of the methods as described in their javadoc comments.

How you choose to internally represent the relation in your class is a design decision that is left up to you. You may also add any other convenience methods that you may find useful, but we will only be testing the methods defined in the skeleton class.

14. **10 points (bonus)** Let $a, b \in \mathbb{Z}$. We say that $a$ is congruent to $b$ modulo $m$ (where $m \geq 2$ is an integer), written

$$ a \equiv b \pmod{m} $$

if

$$ a = k \cdot m + b $$

for some $k \in \mathbb{Z}$. In other words, when you divide $a$ by $m$, the remainder is $b$. For example, $10 \equiv 1 \pmod{3}$ and $10 \equiv 2 \pmod{4}$.

(a) Prove that (for a fixed $m$) a congruence defines an equivalence relation $R$ on the set of all integers $\mathbb{Z}$. That is, the relation

$$(a, b) \in R \iff a \equiv b \pmod{m}$$

is an equivalence relation.

(b) How many equivalence classes are there for $a \equiv b \pmod{m}$?