Instructions Follow all instructions carefully, failure to do so may result in points being deducted. Clearly label each problem and submit the answers in order. Print out a copy of this cover sheet and staple it to the front of your assignment for grading. Be sure to show sufficient work to justify your answers. When asked to prove something, you must give a formal, rigorous, and complete proof. The CSE academic dishonesty policy is in effect (see https://cse.unl.edu/academic-integrity). You are highly encouraged to typeset your assignment using \LaTeX{}; if your answers are not legible, you may be required to use \LaTeX{} in future assignments.

Partner Policy You have the option of working in pairs for this assignment. You always have the option of working alone if you chose. If you choose to work as a pair, you must adhere to these guidelines:

1. You must work on all problems together. You may not simply partition the work between you.

2. You should not discuss problems with other groups or individuals beyond general questions.

3. Hand in only one hard copy with both of your names and logins. The first author should hand in any soft copies of programs.

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1. 4 points Use a proof by contrapositive to prove that if \( x + y \geq 2 \) where \( x, y \in \mathbb{R} \), then \( x \geq 1 \) or \( y \geq 1 \).

2. 5 points (Rosen 1.5.30) Rewrite each of these statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).

1. \( \neg \exists y \exists x P(x, y) \)
2. \( \neg \forall x \exists y P(x, y) \)
3. \( \neg \exists y (Q(y) \land \forall x \neg R(x, y)) \)
4. \( \neg \exists y (\exists x R(x, y) \lor \forall x S(x, y)) \)
5. \( \neg \exists y (\forall x \exists z T(x, y, z) \lor \exists x \forall z U(x, y, z)) \)

3. 5 points (Rosen 1.5.34) Find a common domain for the variables \( x, y, z \) for which the statement

\[ \forall x \forall y [(x \neq y) \rightarrow \forall z ((z = x) \lor (z = y))] \]

is true and another domain for which it is false. Explain your answer.

4. 4 points (Rosen 1.7.16) Prove that if \( m \) and \( n \) are integers and \( mn \) is even, then \( m \) is even or \( n \) is even.

5. 4 points (Rosen 1.7.26) Prove that if \( n \) is a positive integer, then \( n \) is even if and only if \( 7n + 4 \) is even.

6. 5 points (Rosen 1.7.30) Show that these three statements are equivalent, where \( a, b \in \mathbb{R} \): (i) \( a < b \), (ii) the average of \( a, b \), is greater than \( a \), and (iii) the average of \( a \) and \( b \) is less than \( b \).

7. 5 points (Rosen 1.7.32) Show that these statements about the real number \( x \) are equivalent: (i) \( x \) is rational, (ii) \( x/2 \) is rational, (iii) \( 3x - 1 \) is rational. (Note: to show that three statements are equivalent it is enough to show that (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (i); why? Because of hypothetical syllogism!)

8. 7 points (Rosen 2.1.10) Determine whether these statements are true or false.

(a) \( \emptyset \in \{\emptyset\} \)
(b) \( \emptyset \in \{\emptyset, \{\emptyset\}\} \)
(c) \( \{\emptyset\} \in \{\emptyset\} \)
(d) \( \{\emptyset\} \in \{\{\emptyset\}\} \)
(e) \( \{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\} \)
(f) \( \{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\} \)
(g) \( \{\{\emptyset\}\} \subseteq \{\{\emptyset\}, \{\emptyset\}\} \)

9. 4 points Let \( A, B \) be sets and suppose that \( |A| = 5 \), \( |B| = 10 \). Prove or disprove: \( |A \cup B| = 15 \)

10. 8 points Prove or disprove the following set equivalences. Do not use membership tables if you prove.

(a) \( A \setminus B = A \cap \overline{B} \)
(b) \( A \oplus \overline{B} = \overline{A} \oplus B \)

11. The symmetric difference of sets \( A, B \), denoted \( A \oplus B \) is the set containing those elements in either \( A \) or \( B \), but not in both \( A \) and \( B \). You will show that

\[ A \oplus B = (A \setminus B) \cup (B \setminus A) \]

(a) 3 points Prove it using a membership table
(b) **5 points** Prove it without using a membership table, note that \( x \in (A \setminus B) \iff x \in A \land x \notin B \)

12. **6 points** Recall that \( A \times B \) denotes the cartesian product of two sets.
   
   (a) Prove or disprove:
   
   \[
   A \times (B \cap C) = (A \times B) \cap (A \times C)
   \]
   
   (b) Prove or disprove:
   
   \[
   A \times (B \cup C) = (A \times B) \cup (A \times C)
   \]

13. **2 points** Let \( A, B \) be sets such that \(|A| = n, |B| = m\). What is the cardinality of \( \mathcal{P}(A \times B) \)?

14. **4 points** Let \( A^n \) denote the cartesian product of a set \( A \) with itself \( n \) times, that is:
   
   \[
   A^n = A \times A \times \cdots \times A
   \]
   
   (a) What is the cardinality of \( A^n \)?
   
   (b) What is the cardinality of \( \mathcal{P}(A^n) \)?

15. **4 points** Let \( A = \{1, 2, 6\}, B = \{0, 3\} \), and \( C = \{1, 3, 9\} \). Find the following
   
   (a) \( A \times C \)
   
   (b) \( A \times B \times C \)
   
   (c) \( B \times B \times B \)
   
   (d) What would the cardinality of \( A \times B \times C \times B \times A \) be?

16. **50 points** **Programming Assignment** – In this programming assignment, you will implement several algorithms dealing with sets. Specifically, you will implement the methods in the SetUtils Java class provided (see webpage). Note: this class utilizes a Pair utility also provided for you (you need not hand it in). You may add any helper methods you wish, but to simplify grading, ensure that they are generic and static.

17. **10 points** (bonus) In “naive” set theory, you can define a set to contain anything you want. This leads to paradoxes that necessitate the development of more formal axiomatic set theories (implicitly, we’ve been using ZFC). One such paradox (if you are allowed to define a set however you want) is Russell’s paradox. Consider the following set:
   
   \[
   R = \{ x \mid x \notin x \}
   \]
   
   That is, \( R \) is the set of all sets that do not contain themselves as an element. The paradox arises because we’ve defined the membership of a set in terms of itself (a form of self-reference). You’ll explore this paradox further.
   
   (a) Explain why being allow to define \( R \) leads to a paradox by determining whether or not \( R \) is a member of itself, that is, is \( R \in R \)?
   
   (b) How does a programming language like Java handle such set definitions? Explore this by doing the following.
   
   1. Write code to define a Set using raw-types (that is, no parameterization, so the set can hold any type of variable) and add a few strings to it and a few integers to it.
   2. Try adding the set to itself and print out the result (what does the set contain?).
   3. Try removing the set from itself and observe what happens.
   4. Try adding the set to itself a second time and observe what happens.
5. Finally, try using the set’s `contains()` method to test if the set is a member of itself. What happens?

6. In your own words explain the behavior of each of the above operations and whether or not what Java does is the “right” or “correct” thing to do.

Include a print out of your code demonstrating each part above.