Trees

Trees, Binary Search Trees, Heaps & Applications

Dr. Chris Bourke
Department of Computer Science & Engineering
University of Nebraska—Lincoln
Lincoln, NE 68588, USA
cbourke@cse.unl.edu
http://cse.unl.edu/~cbourke

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Abstract

These are lecture notes used in CSCE 156 (Computer Science II), CSCE 235 (Discrete Structures) and CSCE 310 (Data Structures & Algorithms) at the University of Nebraska—Lincoln.
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Part I

Trees

Lecture Outlines

CSCE 156

- Basic definitions
- Tree Traversals

CSCE 235

- Review of basic definitions
- Huffman Coding

CSCE 310

- Heaps, Heap Sort
- Balanced BSTs: 2-3 Trees, AVL, Red-Black

1 Introduction

Motivation: we want a data structure to store elements that offers efficient, arbitrary retrieval (search), insertion, and deletion.

- Array-based Lists
  - $O(n)$ insertion and deletion
  - Fast index-based retrieval
  - Efficient binary search if sorted
- Linked Lists
  - Efficient, $O(1)$ insert/delete for head/tail
- Inefficient, $O(n)$ arbitrary search/insert/delete
- Efficient binary search not possible without random access

- Stacks and queues are efficient, but are restricted access data structures
- Possible alternative: Trees
- Trees have the potential to provide $O(\log n)$ efficiency for all operations

2 Definitions & Terminology

- A tree is an acyclic graph
- For our purposes: a tree is a collection of nodes (that can hold keys, data, etc.) that are connected by edges
- Trees are also oriented: each node has a parent and children
- A node with no parents is the root of the tree, all child nodes are oriented downward
- Nodes not immediately connected can have an ancestor, descendant or cousin relationship
- A node with no children is a leaf
- A tree such that all nodes have at most two children is called a binary tree
- A binary tree is also oriented horizontally: each node may have a left and/or a right child
- Example: see Figure 1
- A path in a tree is a sequence nodes connected by edges
- The length of a path in a tree is the number of edges in the path (which equals the number of nodes in the path minus one)
- A path is simple if it does not traverse nodes more than once (this is the default type of path)
- The depth of a node $u$ is the length of the (unique) path from the root to $u$
- The depth of the root is 0
- The depth of a tree is the maximal depth of any node in the tree (sometimes the term height is used)
• All nodes of the same depth are considered to be at the same level

• A binary tree is complete (also called full or perfect) if all nodes are present at all levels 0 up to its depth $d$

• A sub-tree rooted at a node $u$ is the tree consisting of all descendants with $u$ oriented as the root

![Figure 1: A Binary Tree](image)

Properties:

• In a tree, all nodes are connected by exactly one unique path

• The maximum number of nodes at any level $k$ is $2^k$

• Thus, the maximum number of nodes $n$ for any binary tree of depth $d$ is:

$$n = 2^0 + 2^1 + 2^2 + \cdots + 2^{d-1} + 2^d = \sum_{k=0}^{d} 2^k = 2^{d+1} - 1$$

• Given a full binary tree with $n$ nodes in it has depth:

$$d = \log (n + 1) - 1$$

• That is, $d = O(\log n)$

Motivation: if we can create a tree-based data structure with operations proportional to its depth, then we could potentially have a data structure that allows retrieval/search, insertion, and deletion in $O(\log n)$-time.
3 Tree Traversal

• Given a tree, we need a way to enumerate elements in a tree
• Many algorithms exist to iterate over the elements in a tree
• We’ll look at several variations on a depth-first-search

3.1 Preorder Traversal

• A preorder traversal strategy visits nodes in the following order: root; left-sub-tree; right-sub-tree
• An example traversal on the tree in Figure 1:
  \[a, b, d, g, l, m, r, h, n, e, i, o, c, f, j, p, q, k\]
• Applications:
  – Building a tree, traversing a tree, copying a tree, etc.
  – Efficient stack-based implementation
  – Used in prefix notation (polish notation); used in languages such as Lisp/Scheme

3.2 Inorder Traversal

• An inorder traversal strategy visits nodes in the following order: left-sub-tree; root; right-sub-tree
• An example traversal on the tree in Figure 1:
  \[l, g, r, m, d, h, n, b, e, o, i, a, c, p, j, q, f, k\]
• Applications:
  – Enumerating elements in order in a binary search tree
  – Expression trees

3.3 Postorder Traversal

• A postorder traversal strategy visits nodes in the following order: left-sub-tree; right-sub-tree; root
• An example traversal on the tree in Figure 1:
  \[ l, r, m, g, n, h, d, o, i, e, b, p, q, j, k, f, c, a \]

• Applications:
  – Topological sorting
  – Destroying a tree when manual memory management is necessary (roots are the last thing that get cleaned up)
  – Reverse polish notation (operand-operand-operator, unambiguous, used in old HP calculators)
  – PostScript (Page Description Language)

3.4 Breadth-First Search Traversal

• Breadth-First Search (BFS) traversal is a general graph traversal strategy that explores local or close nodes first before traversing “deeper” into the graph

• When applied to an oriented binary tree, BFS explores the tree level-by-level (top-to-bottom, left-to-right)

3.5 Implementations & Data Structures

• Reference based implementation: TreeNode<T>
  – Owns (through composition) references to: leftChild, rightChild, parent
  – Can use either sentinel nodes or null to indicate missing children and parent

• BinaryTree<T> owns a root

• SVN examples: unl.cse.bst

3.5.1 Preorder Implementations

\[
\text{Algorithm 1: Recursive Preorder Tree Traversal}
\]

Stack-based implementation:
• Initially, we push the tree’s root into the stack
• Within a loop, we pop the top of the stack and process it
• We need to push the node’s children for future processing
• Since a stack is LIFO, we push the right child first.

\[
\begin{align*}
\text{Algorithm 2: Stack-based Preorder Tree Traversal} \\
1 & S \leftarrow \text{empty stack} \\
2 & \text{push } T\text{'s root onto } S \\
3 & \text{while } S \text{ is not empty do} \\
4 & \quad \text{node } \leftarrow S.pop \\
5 & \quad \text{push node’s right-child onto } S \\
6 & \quad \text{push node’s left-child onto } S \\
7 & \quad \text{process node} \\
8 & \text{end}
\end{align*}
\]

3.5.2 Inorder Implementation

Stack-based implementation:

• The same basic idea: push nodes onto the stack as you visit them
• However, we want to delay processing the node until we’ve explored the left-sub-tree
• We need a way to tell if we are visiting the node for the first time or returning from the left-tree exploration
• To achieve this, we allow the node to be null
• If null, then we are returning from a left-tree exploration, pop the top of the stack and process (then push the right child)
• If not null, then we push it to the stack for later processing, explore the left child
**INPUT** : A binary tree, \( T \)

**OUTPUT**: An inorder traversal of the nodes in \( T \)

1. \( S \leftarrow \) empty stack
2. \( u \leftarrow \) root
3. **WHILE** \( S \) is not empty **OR** \( u \neq \) null **DO**
   4. **IF** \( u \neq \) null **THEN**
      5. push \( u \) onto \( S \)
      6. \( u \leftarrow u.\text{leftChild} \)
   7. **ELSE**
      8. \( u \leftarrow S.\text{pop} \)
      9. process \( u \)
     10. \( u \leftarrow u.\text{rightChild} \)
8. **END**
12. **END**

**Algorithm 3**: Stack-based Inorder Tree Traversal

3.5.3 **Postorder Implementation**

Stack-based implementation:

- Same basic ideas, except that we need to distinguish if we’re visiting the node for the first time, second time or last (so that we can process it)
- To achieve this, we keep track of where we came from: a parent, left, or right node
- We keep track of a previous and a current node
**Algorithm 4: Stack-based Postorder Tree Traversal**
3.5.4 BFS Implementation

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>INPUT</strong></td>
<td>A binary tree, ( T )</td>
</tr>
<tr>
<td><strong>OUTPUT</strong>:</td>
<td>A BFS traversal of the nodes in ( T )</td>
</tr>
<tr>
<td>1. ( Q \leftarrow ) empty queue</td>
<td></td>
</tr>
<tr>
<td>2. enqueue ( T )'s root into ( Q )</td>
<td></td>
</tr>
<tr>
<td>3. WHILE ( Q ) is not empty DO</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( node \leftarrow Q).dequeue</td>
</tr>
<tr>
<td></td>
<td>enqueue ( node )'s left-child onto ( Q )</td>
</tr>
<tr>
<td></td>
<td>enqueue ( node )'s right-child onto ( Q )</td>
</tr>
<tr>
<td></td>
<td>print ( node )</td>
</tr>
<tr>
<td>8. END</td>
<td></td>
</tr>
</tbody>
</table>

**Algorithm 5**: Queue-based BFS Tree Traversal

3.5.5 Tree Walk Implementations

- Simple rules based on local information: where you are and where you came from
- No additional data structures required
- Traversal is a “walk” around the perimeter of the tree
- Can use similar rules to determine when the current node should be processed to achieve pre, in, and postorder traversals
- Need to take care with corner cases (when current node is the root or children are missing)
- Pseudocode presented Algorithm 6

3.6 Operations

Basic Operations:

- Search for a particular element/key
- Adding an element
  - Add at most shallow available spot
  - Add at a random leaf
INPUT : A binary tree, $T$
OUTPUT: A Tree Walk around $T$

1. $curr \leftarrow \text{root}$
2. $prevType \leftarrow \text{parent}$
3. WHILE $curr \neq \text{null}$ DO
4. IF $prevType = \text{parent}$ THEN
5. //preorder: process $curr$ here
6. IF $curr.leftChild$ exists THEN
7. //Go to the left child:
8. $curr \leftarrow curr.leftChild$
9. $prevType \leftarrow \text{parent}$
10. ELSE
11. $curr \leftarrow curr$
12. $prevType \leftarrow \text{left}$
13. END
14. ELSE IF $prevType = \text{left}$ THEN
15. //inorder: process $curr$ here
16. IF $curr.rightChild$ exists THEN
17. //Go to the right child:
18. $curr \leftarrow curr.rightChild$
19. $prevType \leftarrow \text{parent}$
20. ELSE
21. $curr \leftarrow curr$
22. $prevType \leftarrow \text{right}$
23. END
24. ELSE IF $prevType = \text{right}$ THEN
25. //postorder: process $curr$ here
26. IF $curr.parent = \text{null}$ THEN
27. //root has no parent, we’re done traversing
28. $curr \leftarrow curr.parent$
29. //are we at the parent’s left or right child?
30. ELSE IF $curr = curr.parent.leftChild$ THEN
31. $curr \leftarrow curr.parent$
32. $prevType \leftarrow \text{left}$
33. ELSE
34. $curr \leftarrow curr.parent$
35. $prevType \leftarrow \text{right}$
36. END
37. END
38. END

Algorithm 6: Tree Walk based Tree Traversal
– Add internally, shift nodes down by some criteria

• Removing elements
  – Removing leaves
  – Removing elements with one child
  – Removing elements with two children

Other Operations:

• Compute the total number of nodes in a tree
• Compute the total number of leaves in a tree
• Given an item or node, compute its depth
• Compute the depth of a tree

4 Binary Search Trees

Regular binary search trees have little structure to their elements; search, insert, delete operations are still linear with respect to the number of tree nodes, \( O(n) \). We want a data structure with operations proportional to its depth, \( O(d) \). To this end, we add structure and order to tree nodes.

• Each node has an associated key

• Binary Search Tree Property: For every node \( u \) with key \( u_k \) in \( T \)
  1. Every node in the left-sub-tree of \( u \) has keys less than \( u_k \)
  2. Every node in the right-sub-tree of \( u \) has keys greater than \( u_k \)

• Duplicate keys can be handled, but you must be consistent and not guaranteed to be contiguous

• Alternatively: do not allow duplicate keys or define a key scheme that ensures a total order

• Inductive property: all sub-trees are also binary search trees

• A full example can be found in Figure 2
4.1 Basic Operations

Observation: a binary search tree has more structure: the key in each node provides information on where a node is not located. We will exploit this structure to achieve $O(\log n)$ operations.

Search/retrieve

- Goal: find a node (and its data) that matches a given key $k$
- Start at the node
- At each node $u$, compare $k$ to $u$’s key, $u_k$:
  - If equal, element found, stop and return
  - If $k < u_k$, traverse to $u$’s left-child
  - If $k > u_k$, traverse to $u$’s right-child
- Traverse until the sub-tree is empty (element not found)
- Analysis: number of comparisons is bounded by the depth of the tree, $O(d)$


**Algorithm 7:** Search algorithm for a binary search tree

**Insert**

- Insert new nodes as leaves
- To determine where it should be inserted: traverse the tree as above
- Insert at the first available spot (first missing child node)
- Analysis: finding the available location is $O(d)$, inserting is just reference juggling, $O(1)$

**Delete**

- Need to first *find* the node $u$ to delete, traverse as above, $O(d)$
- If $u$ is a leaf (no children): its safe to simply delete it
- If $u$ has one child, then we can “promote” it to $u$’s spot ($u$’s parent will now point to $u$’s child)
- If $u$ has two children, we need to find a way to preserve the BST property
  - Want to minimally change the tree’s structure
  - Need the operation to be efficient
  - Find the minimum element of the greater nodes (right sub-tree) or the maximal element of the lesser nodes (left sub-tree)
  - Such an element will have at most one child (which we know how to delete)
- Delete it and store off the key/data
- Replace us key/data with the contents of the minimum/maximum element

- Analysis:
  - Search/Find: $O(d)$
  - Finding the min/max: $O(d)$
  - Swapping: $O(1)$
  - In total: $O(d)$

- Examples illustrated in Figure 3

5 Balanced Binary Search Trees

Motivation:

- Ideally, a full binary tree has depth $d = O(\log n)$
- General BSTs can degenerate such that $d = O(n)$
- Need a way to preserve the BST property while at the same time limiting the depth to $O(\log n)$
- Solution: Balanced Binary Search Trees
- BSTs that efficiently reorganize themselves such that the BST Property is preserved and that the depth is restricted to $O(\log n)$

- Some types of Balanced BSTs:
  - B-trees (and 2-3 trees)
  - AVL Trees
  - Red-Black Trees
  - Splay trees

5.1 2-3 Trees

See 310 Note Set.

5.2 AVL Trees

See 310 Note Set.
(a) A Binary Search Tree

(b) Insertion of a new node (16) into a Binary Search Tree

(c) Deletion of a node with two children (15). (d) Node 15 is replaced with the extremal node, First step: find the maximum node in the left- sub-tree (lesser elements).

(e) Deletion a node with only one child (7). (f) Removal is achieved by simply promoting the single child/subtree.

Figure 3: Binary Search Tree Operations. Figure 3(b) depicts the insertion of (16) into the tree in Figure 3(a). Figures 3(c) and 3(d) depict the deletion of a node (15) with two children. Figures 3(e) and 3(f) depict the deletion of a node with only one child (7).
5.3 Red-Black Trees

See 310 Note Set.

6 Optimal Binary Search Trees

See 310 Note Set.

7 Heaps

Definition 1. A heap is a binary tree that satisfies the following properties.

1. It is a full or complete binary tree: all nodes are present except possibly the last row

2. If the last row is not full, all nodes are full-to-the-left

3. It satisfies the Heap Property: every node has a key that is greater than both of its children (max-heap)

- As a consequence of the Heap Property, the maximal element is always at the root
- Alternatively, we can define a min-heap
- Variations: 2-3 heaps, fibonacci heaps, etc.
- A min-heap example can be found in Figure 4

![Figure 4: A min-heap](image)

Applications
• Heaps are an optimal implementation of a priority queue
• Used extensively in other algorithms (Heap Sort, Prim’s, Dijkstra’s, Huffman Coding, etc.) to ensure efficient operation

7.1 Operations

Insert
• Want to preserve the full-ness property and the Heap Property
• Preserve full-ness: add new element at the end of the heap (last row, first free spot on the left)
• Insertion at the end may violate the Heap Property
• Heapify/fix: bubble up inserted element until Heap Property is satisfied
• Analysis: insert is $O(1)$; heapify: $O(d)$

Remove from top
• Want to preserve the full-ness property and the Heap Property
• Preserve full-ness: swap root element with the last element in the heap (lowest row, right-most element)
• Heap property may be violated
• Heapify/fix: bubble new root element down until Heap Property is satisfied
• Analysis: Swap is $O(1)$; heapify: $O(d)$

Others
• Arbitrary remove
• Find
• Possible, but not ideal: Heaps are restricted-access data structures

Analysis
• All operations are $O(d)$
• Since Heaps are full, $d = O(\log n)$
• Thus, all operations are $O(\log n)$
7.2 Implementations

Array-Based

- Root is located at index 1
- If a node $u$ is at index $i$, $u$’s left-child is at $2i$, its right-child is at $2i + 1$
- If node $u$ is at index $j$, its parent is at index $\lfloor \frac{j}{2} \rfloor$
- Alternatively: 0-index array left/right children/parent are at $2n + 1, 2n + 2, \lfloor \frac{j-1}{2} \rfloor$
- Advantage: easy implementation, all items are contiguous in the array (in fact, a BFS ordering!)
- Disadvantage: Insert operation may force a reallocation, but this can be done in amortized-constant time (though may still have wasted space)

Tree-Based

- Reference-based tree (nodes which contain references to children/parent)
- Parent reference is now required for efficiency
- For efficiency, we need to keep track of the last element in the tree
- For deletes/inserts: we need a way to find the last element and first “open spot”
- We’ll focus on finding the first available open spot as the same technique can be used to find the last element with minor modifications

7.2.1 Finding the first available open spot in a Tree-based Heap

Technique A: numerical technique

- WLOG: assume we keep track of the number of nodes in the heap, $n$ and thus the depth $d = \lfloor \log n \rfloor$
- If $n = 2^{d+1} - 1$ then the tree is full, the last element is all the way to the right, the first available spot is all the way to the left
- Otherwise $n < 2^{d+1} - 1$ and the heap is not full (the first available spot is located at level $d$, root is at level 0)
- Starting at the root, we want to know if the last element is in the left-subtree or the right subtree
• Let \( m = n - (2^d - 1) \), the number of nodes present in level \( d \)

• If \( m \geq \frac{2^d}{2} \) then the left-sub tree is full at the last level and so the next open spot would be in the right-sub tree

• Otherwise if \( m < \frac{2^d}{2} \) then the left-sub tree is not full at the last level and so the next open spot is in the left-sub tree

• Traverse down to the left or right respectively and repeat: the resulting sub-tree will have depth \( d - 1 \) with \( m = m \) (if traversing left) or \( m = m - \frac{2^d}{2} \) (if traversing right)

• Repeat until we’ve found the first available spot

• Analysis: in any case, its \( O(d) = O(\log n) \) to traverse from the root to the first open spot
INPUT: A tree-based heap $H$ with $n$ nodes

OUTPUT: The node whose child is the next available open spot in the heap

1. $curr \leftarrow T$.head
2. $d \leftarrow \lfloor \log n \rfloor$
3. $m \leftarrow n$
4. WHILE $curr$ has both children DO
   5. IF $m = 2^{d+1} - 1$ THEN
      //remaining tree is full, traverse all the way left
      6. WHILE $curr$ has both children DO
         7. $curr \leftarrow curr$.leftChild
      END
   ELSE
      //remaining tree is not full, determine if the next open spot is in the left or right sub-tree
      IF $m \geq \frac{2^d}{2}$ THEN
         //left sub-tree is full
         10. $d \leftarrow (d - 1)$
         11. $m \leftarrow (m - \frac{2^d}{2})$
         12. $curr \leftarrow curr$.rightChild
      ELSE
         //left sub-tree is not full
         15. $d \leftarrow (d - 1)$
         16. $m \leftarrow m$
         17. $curr \leftarrow curr$.leftChild
      END
   END
5. END
6. output $curr$

**Algorithm 8:** Find Next Open Spot - Numerical Technique

Technique B: Walk Technique

- Alternatively, we can adapt the idea behind the tree walk algorithm to find the next available open spot
- We’ll assume that we’ve kept track of the last node
- If the tree is full, we simply traverse all the way to the left and insert, $O(d)$
- If the last node is a left-child then its parent’s right child is the next available spot, finding it is $O(1)$
- Otherwise, we’ll need to traverse around the perimeter of the tree until we reach the next open slot

```
INPUT : A tree-based heap $H$ with $n$ nodes
OUTPUT: The node whose (missing) child is the next available open spot in the heap

1  $d \leftarrow \lfloor \log n \rfloor$
2  IF $n = 2^{d+1} - 1$ THEN
3      //The tree is full, traverse all the way to the left
4   curr $\leftarrow$ root
5   WHILE curr.leftChild $\neq$ null DO
6       curr $\leftarrow$ curr.leftChild
7   END
8  ELSE IF last is a left-child THEN
9      //parent’s right child is open
10     curr $\leftarrow$ last.parent
11  ELSE
12      //The open spot lies in a subtree to the right of the last node
13      //Walk the tree until we reach it
14     curr $\leftarrow$ last.parent
15     WHILE curr is a right-child DO
16         curr $\leftarrow$ curr.parent
17     END
18     //"turn" right
19     curr $\leftarrow$ curr.parent
20     curr $\leftarrow$ curr.rightChild
21     //traverse all the way left
22     WHILE curr.leftChild $\neq$ null DO
23         curr $\leftarrow$ curr.leftChild
24     END
25  END
26  //current node’s missing child is the open spot
27  output curr
```

**Algorithm 9:** Find Next Open Spot - Walk Technique
7.3 Heap Sort

- If min/max element is always at the top; simply insert all elements, then remove them all!
- Perfect illustration of “Smart data structures and dumb code are a lot better than the other way around”

```
INPUT : A collection of elements $A = \{a_1, \ldots, a_n\}$
OUTPUT: A collection, $A'$ of elements in $A$, sorted
1 $H \leftarrow$ empty heap
2 $A' \leftarrow$ empty collection
3 FOREACH $x \in A$ DO
4 | insert $x$ into $H$
5 END
6 WHILE $H$ is not empty DO
7 | $y \leftarrow$ remove top from $H$
8 | Add $y$ to the end of $A'$
9 END
10 output $A'$
```

**Algorithm 10: Heap Sort**

Analysis

- Amortized analysis: insert/remove operations are not constant throughout the algorithm
- On first iteration: insert is $d = O(1)$; on the $i$-th iteration, $d = O(\log i)$; only on the last iteration is insertion $O(\log n)$
- In total, the insert phase is:
  \[
  \sum_{i=1}^{n} \log i = O(n \log n)
  \]
- A similar lower bound can be shown
- Same analysis applies to the remove phase:
  \[
  \sum_{i=n}^{1} \log i
  \]
- In total, $O(n \log n)$
8 Java Collections Framework

Java has support for several data structures supported by underlying tree structures.

- `java.util.PriorityQueue<E>` is a binary-heap based priority queue
  - Priority (keys) based on either natural ordering or a provided Comparator
  - Guaranteed $O(\log n)$ time for insert (offer) and get top (poll)
  - Supports $O(n)$ arbitrary remove(Object) and search (contains) methods

- `java.util.TreeSet<E>`
  - Implements the SortedSet interface; makes use of a Comparator
  - Backed by TreeMap, a red-black tree balanced binary tree implementation
  - Guaranteed $O(\log n)$ time for add, remove, contains operations
  - Default iterator is an in-order traversal

9 Applications

9.1 Huffman Coding

Overview

- Coding Theory is the study and theory of codes—schemes for transmitting data
- Coding theory involves efficiently padding out data with redundant information to increase reliability (detect or even correct errors) over a noisy channel
- Coding theory also involves compressing data to save space
  - MP3s (uses a form of Huffman coding, but is information lossy)
  - jpegs, mpegs, even DVDs
  - pack (straight Huffman coding)
  - zip, gzip (uses a Ziv-Lempel and Huffman compression algorithm)

Basics

- Let $\Sigma$ be a fixed alphabet of size $n$
- A coding is a mapping of this alphabet to a collection of binary codewords,
  \[ \Sigma \rightarrow \{0,1\}^* \]
• A *block encoding* is a *fixed length encoding* scheme where all codewords have the same length (example: ASCII); requires ⌈log₂ n⌉ length codes

• Not all symbols have the same frequency, alternative: *variable length encoding*

• Intuitively: assign shorter codewords to more frequent symbols, longer to less frequent symbols

• Reduction in the overall average codeword length

• Variable length encodings must be *unambiguous*

• Solution: *prefix free codes*: a code in which no whole codeword is the prefix of another (other than itself of course).

• Examples:
  
  – \{0, 01, 101, 010\} is not a prefix free code.
  
  – \{10, 010, 110, 0110\} is a prefix free code.

• A simple way of building a prefix free code is to associate codewords with the *leaves* of a binary tree (not necessarily full).

• Each edge corresponds to a bit, 0 if it is to the left sub-child and 1 to the right sub-child.

• Since no simple path from the root to any leaf can continue to another leaf, then we are guaranteed a prefix free coding.

• Using this idea along with a greedy encoding forms the basis of *Huffman Coding*

**Steps**

• Consider a precomputed relative frequency function:

  \[
  \text{freq} : \Sigma \rightarrow [0, 1]
  \]

• Build a collection of *weighted* trees \(T_x\) for each symbol \(x \in \Sigma\) with \(wt(T_x) = \text{freq}(x)\)

• Combine the two least weighted trees via a new node; associate a new weight (the sum of the weights of the two subtrees)

• Keep combining until only one tree remains

• The tree constructed in Huffman’s algorithm is known as a *Huffman Tree* and it defines a *Huffman Code*
Algorithm 11: Huffman Coding

9.1.1 Example

Construct the Huffman Tree and Huffman Code for a file with the following content.

<table>
<thead>
<tr>
<th>character</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency</td>
<td>0.10</td>
<td>0.15</td>
<td>0.34</td>
<td>0.05</td>
<td>0.12</td>
<td>0.21</td>
<td>0.03</td>
</tr>
</tbody>
</table>

- Average codeword length:
  \[0.10 \cdot 3 + 0.15 \cdot 3 + 0.34 \cdot 2 + 0.05 \cdot 4 + 0.12 \cdot 3 + 0.21 \cdot 2 + 0.03 \cdot 4 = 2.53\]

- Compression ratio:
  \[\frac{3 - 2.53}{3} = 15.67\%\]

- In general, for text files, pack (Huffman Coding), claims an average compression ratio of 25-40%.
- Degenerative cases:
Figure 5: Huffman Tree

- When the probability distribution is uniform: \( p(x) = p(y) \) for all \( x, y \in \Sigma \)
- When the probability distribution follows a fibonacci sequence (the sum of each of the two smallest probabilities is less than equal to the next highest probability for all probabilities)

A Stack-based Traversal Simulations

A.1 Preorder

Simulation of a stack-based preorder traversal of the binary tree in Figure 1.

```
push a
(enter loop)
pop, node = a
push c
push b
print a
(enter loop)
pop, node = b
push e
push d
print b
(enter loop)
pop, node = d
push h
push g
print d
(enter loop)
pop, node = m
push r
print m
(enter loop)
pop, node = c
push f
(enter loop)
pop, node = e
push i
(enter loop)
pop, node = f
push k
(enter loop)
pop, node = j
push q
(enter loop)
pop, node = p
push q
(enter loop)
pop, node = o
push p
(enter loop)
pop, node = n
push o
(enter loop)
pop, node = o
push o
(enter loop)
pop, node = o
push o
(enter loop)
pop, node = o
push o
(enter loop)
pop, node = o
push o
(enter loop)
pop, node = o
push o
(enter loop)
pop, node = o
push o
(enter loop)
pop, node = o
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pop, node = o
push o
(enter loop)
pop, node = o
push o
(enter loop)
pop, node = o
push o
(enter loop)
pop, node = o
push o
(enter loop)
pop, node = o
push o
(enter loop)
pop, node = o
push o
(enter loop)
pop, node = o
push o
(enter loop)
pop, node = o
```
### A.2 Inorder

Simulation of a stack-based inorder traversal of the binary tree in Figure 1.

```
(no push)  (enter loop)  (no push)  (enter loop)  (no push)
print p    pop, node = q  print q    pop, node = k  print k

u = a
push a
update u = b
push b
update u = d
push d
update u = g
push g
update u = l
push l
update u = null
push l
update u = null
push h
update u = null
push h
update u = null
push o
update u = null
push o
update u = n
push n
update u = null
push r
update u = r
push r
update u = null

u = b
pop b, update
process b
update u = e
pop e, update
process e
update u = i
pop i, update
process i
update u = null

u = m
push m, update
process m
update u = null

u = r
pop r, update
process r
update u = null

u = null
process u = null
update u = null

u = null
process u = null
update u = null

u = a
process u = a
update u = c
update u = null

u = i
push i
update u = null

u = null
process u = null
update u = null

u = c
push c, update
process c
update u = f
update u = null

u = b
push b, update
process b
update u = null

u = null
process u = null
update u = null

u = p
process u = p
update u = null

u = q
push q
update u = q

u = null
process u = null
update u = null

u = k
push k
update u = k

u = null
process u = null
update u = null

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```
### A.3 Postorder

Simulation of a stack-based postorder traversal of the binary tree in Figure 1:

<table>
<thead>
<tr>
<th>prev = null</th>
<th>process l</th>
<th>update prev = l</th>
<th>update prev = m</th>
<th>update curr = (n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>push a</td>
<td>(enter loop)</td>
<td>(enter loop)</td>
<td>(enter loop)</td>
<td>(enter loop)</td>
</tr>
<tr>
<td>update curr = (a)</td>
<td>update curr = (m)</td>
<td>update curr = (m)</td>
<td>update curr = (n)</td>
<td>check:</td>
</tr>
<tr>
<td>check: prev = null</td>
<td>check: prev.rightChild = curr</td>
<td>check: prev.rightChild = curr</td>
<td>check: prev.rightChild = curr</td>
<td>check:</td>
</tr>
<tr>
<td>push (b)</td>
<td>push (g)</td>
<td>push (g)</td>
<td>(null.rightChild = (g))</td>
<td>(null.rightChild = (g))</td>
</tr>
<tr>
<td>update prev = a</td>
<td>update prev = m</td>
<td>update prev = m</td>
<td>update prev = n</td>
<td>process m</td>
</tr>
<tr>
<td>(enter loop)</td>
<td>(enter loop)</td>
<td>(enter loop)</td>
<td>(enter loop)</td>
<td>(enter loop)</td>
</tr>
<tr>
<td>update curr = (b)</td>
<td>update curr = (m)</td>
<td>update curr = (h)</td>
<td>update curr = (d)</td>
<td>check:</td>
</tr>
<tr>
<td>check: prev.leftChild = curr</td>
<td>check: prev.rightChild = curr</td>
<td>check: prev.rightChild = curr</td>
<td>check: prev.rightChild = curr</td>
<td>check:</td>
</tr>
<tr>
<td>((b).leftChild = (b))</td>
<td>((l).leftChild = (l))</td>
<td>(null.rightChild = (h))</td>
<td>(null.rightChild = (d))</td>
<td>process h</td>
</tr>
<tr>
<td>push (d)</td>
<td>push (m)</td>
<td>push (d)</td>
<td>update prev = d</td>
<td>update prev = d</td>
</tr>
<tr>
<td>update prev = b</td>
<td>update prev = m</td>
<td>update prev = m</td>
<td>update prev = d</td>
<td>update prev = d</td>
</tr>
<tr>
<td>(enter loop)</td>
<td>(enter loop)</td>
<td>(enter loop)</td>
<td>(enter loop)</td>
<td>(enter loop)</td>
</tr>
<tr>
<td>update curr = (d)</td>
<td>update curr = (m)</td>
<td>update curr = (b)</td>
<td>update curr = (d)</td>
<td>check:</td>
</tr>
<tr>
<td>check: prev.leftChild = curr</td>
<td>check: prev.rightChild = curr</td>
<td>check: prev.rightChild = curr</td>
<td>check: prev.rightChild = curr</td>
<td>check:</td>
</tr>
<tr>
<td>((d).leftChild = (d))</td>
<td>((m).rightChild = (m))</td>
<td>((m).rightChild = (d))</td>
<td>((n).rightChild = (d))</td>
<td>process d</td>
</tr>
<tr>
<td>push (g)</td>
<td>push (r)</td>
<td>push (r)</td>
<td>(null.rightChild = (r))</td>
<td>(null.rightChild = (r))</td>
</tr>
<tr>
<td>update prev = d</td>
<td>update prev = r</td>
<td>update prev = r</td>
<td>update prev = h</td>
<td>update prev = h</td>
</tr>
<tr>
<td>(enter loop)</td>
<td>(enter loop)</td>
<td>(enter loop)</td>
<td>(enter loop)</td>
<td>(enter loop)</td>
</tr>
<tr>
<td>update curr = (g)</td>
<td>update curr = (r)</td>
<td>update curr = (h)</td>
<td>update curr = (e)</td>
<td>check:</td>
</tr>
<tr>
<td>check: prev.leftChild = curr</td>
<td>check: prev.rightChild = curr</td>
<td>check: prev.rightChild = curr</td>
<td>check: prev.rightChild = curr</td>
<td>check:</td>
</tr>
<tr>
<td>((g).leftChild = (g))</td>
<td>((g).rightChild = (g))</td>
<td>((h).rightChild = (h))</td>
<td>((e).rightChild = (e))</td>
<td>process e</td>
</tr>
<tr>
<td>push (l)</td>
<td>push (h)</td>
<td>push (n)</td>
<td>(noop)</td>
<td>(noop)</td>
</tr>
<tr>
<td>update prev = g</td>
<td>update prev = d</td>
<td>update prev = h</td>
<td>update prev = e</td>
<td>update prev = e</td>
</tr>
<tr>
<td>(enter loop)</td>
<td>(enter loop)</td>
<td>(enter loop)</td>
<td>(enter loop)</td>
<td>(enter loop)</td>
</tr>
<tr>
<td>update curr = (l)</td>
<td>update curr = (m)</td>
<td>update curr = (n)</td>
<td>update curr = (i)</td>
<td>check:</td>
</tr>
<tr>
<td>check: prev.leftChild = curr</td>
<td>check: prev.rightChild = curr</td>
<td>check: prev.rightChild = curr</td>
<td>check: prev.rightChild = curr</td>
<td>check:</td>
</tr>
<tr>
<td>((l).leftChild = (l))</td>
<td>((null.rightChild = (r))</td>
<td>((null.rightChild = (m))</td>
<td>(null.rightChild = (m))</td>
<td>(null.rightChild = (m))</td>
</tr>
<tr>
<td>(noop)</td>
<td>process r</td>
<td>process m</td>
<td>process n</td>
<td>process n</td>
</tr>
<tr>
<td>update prev = l</td>
<td>update prev = r</td>
<td>update prev = h</td>
<td>update prev = e</td>
<td>update prev = e</td>
</tr>
<tr>
<td>(enter loop)</td>
<td>(enter loop)</td>
<td>(enter loop)</td>
<td>(enter loop)</td>
<td>(enter loop)</td>
</tr>
<tr>
<td>update curr = (l)</td>
<td>update curr = (l)</td>
<td>update curr = (i)</td>
<td>update curr = (i)</td>
<td>check:</td>
</tr>
<tr>
<td>check: prev.rightChild = curr</td>
<td>check: curr.leftChild = prev</td>
<td>check: curr.leftChild = prev</td>
<td>check: curr.leftChild = prev</td>
<td>check:</td>
</tr>
<tr>
<td>(null.rightChild = (l))</td>
<td>((r).leftChild = (r))</td>
<td>((r).leftChild = (r))</td>
<td>((r).leftChild = (r))</td>
<td>(null.rightChild = (l))</td>
</tr>
</tbody>
</table>
((i).rightChild = (i))

push (o)

update prev = i

(enter loop)

update curr = (o)

check:

prev.leftChild = curr

((o).leftChild = (a))

(noop)

update prev = o

(enter loop)

update curr = (a)

check:

prev.rightChild = curr

((a).rightChild = (a))

process o

update prev = o

(enter loop)

update curr = (i)

check:

prev.rightChild = curr

((null.rightChild = (i))

check:

curr.leftChild = prev

((0).leftChild = (a))

update prev = i

(enter loop)

update curr = (i)

check:

prev.rightChild = curr

((null.rightChild = (i))

process i

update prev = i

(enter loop)

update curr = (e)

check:

prev.rightChild = curr

((null.rightChild = (e))

process e

update prev = e

(enter loop)

update curr = (b)

update prev = p

(((q).rightChild = (f))

check:

curr.leftChild = prev

((j).leftChild = (j))

update curr = (p)

check:

prev.rightChild = curr

((null.rightChild = (p))

process p

update prev = p

(enter loop)

update curr = (k)

check:

curr.rightChild = curr

((null.rightChild = (k))

(noop)

update prev = k

(enter loop)

update curr = (k)

check:

prev.rightChild = curr

((null.rightChild = (k))

process k

update prev = k

(enter loop)

update curr = (q)

check:

prev.rightChild = curr

((null.rightChild = (q))

(noop)

update prev = q

(enter loop)

update curr = (q)

check:

prev.rightChild = curr

((null.rightChild = (q))

process q

update prev = q

(enter loop)

update curr = (q)

check:

prev.rightChild = curr

((null.rightChild = (q))

process q

update prev = q

(enter loop)

update curr = (c)

check:

prev.rightChild = curr

((null.rightChild = (c))

process c

update prev = c

(enter loop)

update curr = (a)

check:

prev.rightChild = curr

((null.rightChild = (a))

(process a

update prev = a

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