Homework 2

Due on 25th March, 2008

Answers should be presented clearly and legibly. You are strongly advised to use some document preparation system for typesetting the answers.

1. **NP-completeness.**

   (a) (10 Points) Let $G = (V, E)$ be a *directed* graph. A set $F \subseteq V$ is a feedback vertex set if every cycle of G contains a vertex in F. The FEEDBACK VERTEX SET problem asks whether G has a feedback vertex set with at most K vertices. Show that FEEDBACK VERTEX SET is NP-complete. (*Hint:* You may assume that Vertex Cover problem is NP-complete and use it for reduction.)

   (b) (10 Points) Given an $m \times n$ matrix A and an integer m-vector b, the 0-1 INTEGER PROGRAMMING problem (0-1-IP) asks whether there is an integer n-vector x with elements in the set $\{0, 1\}$ such that $Ax \leq b$. Prove that 0-1-IP is NP-complete. (*Hint: Give a reduction from* 3SAT).

2. **Monotone CVP** (20 Points). In the lecture we showed that CIRCUIT VALUE PROBLEM (CVP), where the circuit can have AND, OR, and NOT gates, is P-complete. Consider a restricted version of CVP where the encoded circuit does not have NOT-gates; it has only AND and OR gates. Such a circuit can only compute monotone Boolean functions. Show that this MONOTONE CVP is also P-complete. (FYI: Monotone Boolean functions are boolean functions which has the property that if one of the inputs changes from False to True, the value of the function *cannot* change from True to False.) (*Hint: Reduce from CVP. First move all internal NOT gates to the bottom of the circuit.*)

3. **Stratified CVP** (20 Points). A Boolean circuit is stratified if it either contains only AND-gates or only OR-gates, but not both, and it does not contain NOT-gates. Show that the STRATIFIED CVP, circuit value problem restricted to stratified Boolean circuits, is NL-complete.

4. (20 Points) Majority function on n Boolean variables is defined as

$$\begin{aligned} \textsc{Maj}(x_1, x_2, \ldots, x_n) &= 1 \quad \text{if } \Sigma_{i=1}^{n} x_i \geq \lceil n/2 \rceil \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

   Design a Boolean circuit of size $O(n)$ for computing MAJ.

5. (20 Points). Show that any Boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$ has a circuit of size 2^n. (*Hint*: $f(x_1, \cdots, x_n) = x_1 f(1, x_2, \cdots x_n) \vee \overline{x} f(0, x_2, \cdots x_n)$)