CSCE 428/828 HW 5

due via handin Monday 4/24/2017
1. (25 points) Prove that the set of rational number is **countably infinite**.

**SOLUTION:** See Example 4.15 in the text. The solution shows that we can have a correspondence \( f \) that maps (positive) rational numbers to natural numbers as follows: \( \frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{1}{3}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \ldots \mapsto 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots \) To consider 0 and negative rational numbers, we can insert these after the corresponding positive rational ones as follows: \( 0/1, 1/1, -1/1, 2/1, -2/1, 1/2, -1/2, \ldots \).
2. (25 points) Prove that language $HALT_{LBA}$ is decidable by constructing a decider for the language.

$$HALT_{LBA} = \{< L, w > \mid L \text{ is an LBA and } L \text{ halts on string } w.\}$$

**PROOF:**

We construct a decider $R$ with input $< L, w >$ for language $HALT_{LBA}$ as follows.

On input $< L, w >$

- $R$ simulates $L$ on input $w$ for at most $qng^n$ steps
  - If $L$ halts and accepts $w$, then $R$ accepts $< L, w >$;
  - if $L$ halts and rejects $w$, then $R$ accepts $< L, w >$;
  - if $L$ does not halt within $qng^n$ steps, then $R$ halts and rejects $< L, w >$. 


3. (25 points) Prove that language $F_{TM}$ is undecidable by mapping reducing $A_{TM}$ to language $F_{TM}$.

$$F_{TM} = \{ < M_1 > \mid \text{TM } M_1 \text{ halts on input string “Spring” and loops on input string “2017”} \}$$

**PROOF:**

Proof by contradiction. Assume that $F_{TM}$ is decidable, and thus exists a decider $D$ for $F_{TM}$. We can then use $D$ to construct a decider $S$ for $A_{TM}$ (shown below). But this is a contradiction because $A_{TM}$ is undecidable, thus our assumption is incorrect and that $F_{TM}$ is undecidable.

$S$ takes as input $< M, w >$:

- create a TM $M_1$ using by applying the map reduction function $f$ on $< M, w >$ (described below)
- run $D$ on $M_1$
  - if $D$ accepts, then $S$ accepts
  - if $D$ rejects, then $S$ rejects

Computable function $f$ constructs a special TM $M_1$ so that

$$< M, w > \in A_{TM} \iff < M_1 > \in F_{TM}$$

More specifically, $f$ does the following mapping.

<table>
<thead>
<tr>
<th>string $&lt; M, w &gt;$</th>
<th>string $&lt; M_1 &gt;$</th>
</tr>
</thead>
<tbody>
<tr>
<td>in the language</td>
<td>$M$ accepts $w$ $\iff$ $M_1$ halts on “Spring” and loops on “2017”</td>
</tr>
<tr>
<td>not in the language</td>
<td>$M$ rejects $w$ $\iff$ $M_1$ loops on “Spring” or halts on “2017” invalid encoding</td>
</tr>
</tbody>
</table>

TM $M_1$ works as follows on input string $x$.

- $M_1$ simulates $M$ on $w$
- if $M$ accepts $w$, then
  - if $x$ is “Spring”, then $M_1$ accepts/rejects $x$, and if $x$ is “2017”, then $M_1$ loops on $x$.
  - otherwise, it does not matter what $M_1$ does.
- if $M$ rejects $w$, then
  - if $x$ is “Spring”, then $M_1$ loops on $x$, or if $x$ is “2017”, then $M_1$ accepts/rejects $x$.
  - otherwise, it does not matter what $M_1$ does.
- if $M$ loops on $w$, then $M_1$ loops on every input string.
4. (25 points) Show that language $E_{TM}$ is Turing unrecognizable by mapping reducing $\overline{A_{TM}}$ to $E_{TM}$.

$$E_{TM} = \{ <M_1> | M_1 \text{ is a TM, and } L(M_1) = \emptyset \}$$

**PROOF:**

For the purpose of contradiction, we assume that $E_{TM}$ is Turing recognizable, and there exists a TM $R$ for $E_{TM}$. We construct a TM $S$ for $\overline{A_{TM}}$ as follows.

Mapping function $f$ constructs a special Turing machine $M_1$, so that

$$<M, w> \in \overline{A_{TM}} \iff <M_1> \in E_{TM}$$

<table>
<thead>
<tr>
<th>String $&lt;M, w&gt;$</th>
<th>String $&lt;M_1&gt;$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$ rejects $w$</td>
<td>$L(M_1) = \emptyset$</td>
</tr>
<tr>
<td>$M$ loops on $w$</td>
<td><strong>invalid encoding</strong></td>
</tr>
<tr>
<td>$M$ accepts $w$</td>
<td>$L(M_1) = \Sigma^*$</td>
</tr>
<tr>
<td><strong>invalid encoding</strong></td>
<td></td>
</tr>
</tbody>
</table>

![Diagram](image)