Counterexample-guided Approach to Finding Numerical Invariants

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**Introduction**

*Invariants are asserted properties, such as relations among variables that always hold at certain locations in a program*

- Assertions
- Pre/Post conditions
- Loop invariants
Introduction

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- Pre/Post conditions
- Loop invariants

Techniques for automatic invariant generation

- Static: examine program code, compute sound results, but can be expensive and limited to simple invariants
- Dynamic: analyze exec traces, produce expressive invariants, but unsound
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- Static: examine program code, compute sound results, but can be expensive and limited to simple invariants
- Dynamic: analyze exec traces, produce expressive invariants, but unsound

Numerical invariants, e.g., relations among numerical variables

- E.g., \( x = 2y + 3, 0 \leq idx \leq |arr| - 1, x \leq y^2, x = qy + r \)
- Nonlinear polynomial invariants: \( x \leq y^2, x = qy + r, \ldots \)
Invariants can help understanding programs

int cohendiv(int x, int y){
    assert(x>0 && y>0);
    int q=0; int r=x;
    while(r ≥ y){
        int a=1;
        int b=y;
        while[L1](r ≥ 2*b){
            a = 2*a;
            b = 2*b;
        }
        r=r-b;
        q=q+a;
    }
    [L2]
    return q;
}
Invariants can help understanding programs

```c
int cohendiv(int x, int y){
    assert(x>0 && y>0);
    int q=0; int r=x;
    while(r ≥ y){
        int a=1;
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    }
    [L2]
    return q;
}
```

What does this program do? What properties hold at \textbf{L1} and \textbf{L2}?

- **loop invariants at \textbf{L1}:**
  - \( x = qy + r \)
  - \( b = ya \)
  - \( y \leq b \)
  - \( b \leq r \)
  - \( r \leq x \)
  - \( a \leq b \)
  - \( 2 \leq a + y \)

- **postconditions at \textbf{L2}:**
  - \( x = qy + r \)
  - \( 1 \leq q + r \)
  - \( r \leq y - 1 \)
  - \( 0 \leq r \)
  - \( r \leq x \)
Invariants can help understanding programs

```c
int cohendiv(int x, int y){
    assert(x>0 && y>0);
    int q=0; int r=x;
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            b = 2*b;
        }
        r=r-b;
        q=q+a;
    }
    [L2]
    return q;
}
```

What does this program do? What properties hold at L1 and L2?

- **loop invariants at L1:**
  
  \[
  \begin{align*}
  x &= qy + r & b &= ya \\
  y &≤ b & b &≤ r \\
  r &≤ x & a &≤ b \\
  2 &≤ a + y
  \end{align*}
  \]

- **postconditions at L2:**
  
  \[
  \begin{align*}
  x &= qy + r \\
  1 &≤ q + r & r &≤ y - 1 \\
  0 &≤ r & r &≤ x
  \end{align*}
  \]

Describe the semantic the program (e.g., \( x = qy + r \) for integer division) and reveal useful information (e.g., remainder \( r \) is non-negative)
Invariants can help analyze program complexities

```c
void triple(int M, int N, int P){
    assert (0 <= M);
    assert (0 <= N);
    assert (0 <= P);
    int i = 0, j = 0, k = 0;
    int t = 0;
    while(i < N){
        j = 0; t++;
        while(j < M){
            j++; k = i; t++;
            while (k < P){
                k++; t++;
            }
            i = k;
        }
        i++;
    }
    [L]
}
```

Complexity of this program?

- Use \( t \) to count loop iterations

At first glance:

\[ t = O(MNP) \]

A more precise complexity bound:

\[ t = O(N + NM + P) \]

Both are nonlinear invariants
Invariants can help analyze program complexities

void triple(int M, int N, int P){
    assert (0 <= M);
    assert (0 <= N);
    assert (0 <= P);
    int i = 0, j = 0, k = 0;
    int t = 0;
    while(i < N){
        j = 0; t++;
        while(j < M){
            j++; k = i; t++;
            while (k < P){
                k++; t++;
            }
            i = k;
        }
        i++;
    }
    [L]
}
Invariants can help verify programs

```c
void f(int u1, int u2) {
    assert(u1 > 0 && u2 > 0);
    int a = 1, b = 1, c = 2, d = 2;
    int x = 3, y = 3;
    int i1 = 0, i2 = 0;
    while (i1 < u1) {
        i1++;
        x = a + c; y = b + d;
        if ((x + y) % 2 == 0) {
            a++; d++;
        } else { a--;}
    }
    i2 = 0;
    while (i2 < u2 ) {
        i2++;
        c--; b--;
    }
    [L]
    assert(a + c == b + d);
}
```

```c
void g(int n, int u1) {
    assert(u1 > 0);
    int x = 0;
    int m = 0;
    while (x < n) {
        if (u1) {
            m = x;
        }
        x = x + 1;
    }
    [L]
    if (n > 0){
        assert(0 <= m && m < n);
    }
}
```
Invariants can help verify programs

```c
void f(int u1, int u2) {
    assert(u1 > 0 && u2 > 0);
    int a = 1, b = 1, c = 2, d = 2;
    int x = 3, y = 3;
    int i1 = 0, i2 = 0;
    while (i1 < u1) {
        i1++;
        x = a + c; y = b + d;
        if ((x + y) % 2 == 0) {
            a++; d++;
        } else {
            a--;
        }
        i2 = 0;
        while (i2 < u2) {
            i2++;
            c--; b--;
        }
    }
    [L]
    assert(a + c == b + d);
}
void g(int n, int u1) {
    assert(u1 > 0);
    int x = 0;
    int m = 0;
    while (x < n) {
        if (u1) {
            m = x;
        }
        x = x + 1;
    }
    [L]
    if (n > 0){
        assert(0 <= m && m < n);
    }
}
```

Assertions hold if matched or implied by discovered invariants at L
Focus on polynomial invariants over numerical variables

Use CounterExample-Guided Invariant Generation (CEGIR) approach

- Dynamic Inference: use DIG’s algorithms to infer nonlinear equalities and linear inequalities from traces
- Static Checking: use KLEE to check candidate invariants and generate counterexample inputs
Example: Dynamic Inference using DIG

```c
int cohendiv(int x, int y){
    assert(x>0 ; y>0);
    int q=0; int r=x;
    while(r >= y){
        int a=1; int b=y;
        while[L1](r >= 2*b){
            a = 2*a; b = 2*b;
        }
        r=r-b; q=q+a;
    }
    return q;
}
```

Traces:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>a</th>
<th>b</th>
<th>q</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>15</td>
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<td>1</td>
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<td>2</td>
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<td>4</td>
</tr>
</tbody>
</table>

Loop invariants at L1:

Equations:

\[ x = qy + r \]

\[ b = ya \]

Inequalities:

\[ 2 \leq a + y \]

\[ a \leq b \]

\[ b \leq r \]

\[ r \leq x \]


```c
int cohendiv(int x, int y){
    assert(x>0 ; y>0);
    int q=0; int r=x;
    while(r >= y){
        int a=1; int b=y;
        while[L1](r >= 2*b){
            a = 2*a; b = 2*b;
        }
        r=r-b; q=q+a;
    }
    return q;
}
```

Traces:

<p>| | | | | |</p>
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</tr>
</tbody>
</table>

Loop invariants at L1:

- Equations:
  - \( x = qy + r \)
  - \( b = ya \)
- Inequalities:
  - \( 2 \leq a + y \)
  - \( y \leq b \)
  - \( b \leq r \)
  - \( r \leq x \)

7
Example: Dynamic Inference using DIG

```c
int cohendiv(int x, int y){
    assert(x>0 ; y>0);
    int q=0; int r=x;
    while(r >= y){
        int a=1; int b=y;
        while[R1](r >= 2*b){
            a = 2*a; b = 2*b;
        }
        r=r-b; q=q+a;
    }
    return q;
}
```

Traces:
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</table>
```

Loop invariants at **L1:**

- equations: $x = qy + r$, $b = ya$
- inequalities: $2 \leq a + y$, $a \leq b$, $y \leq b$, $b \leq r$, $r \leq x$
Infer Nonlinear Equations using Equation Solver

Terms and degrees

\[ V = \{ r, y, a \} \]
\[ \deg = 2 \]

\[ T = \{ 1, r, y, a, ry, ra, ya, r^2, y^2, a^2 \} \]

Nonlinear equation template

\[ c_1 + c_2 r + c_3 y + c_4 a + c_5 ry + c_6 ra + c_7 ya + c_8 r^2 + c_9 y^2 + c_{10} a^2 = 0 \]

System of linear equations

\[ \text{trace 1} \rightarrow \{ r = 15, y = 2, a = 1 \} \]
\[ \text{eq 1} \rightarrow c_1 + 15 c_2 + 2 c_3 + c_4 + 30 c_5 + 15 c_6 + 2 c_7 + 225 c_8 + 4 c_9 + c_{10} = 0 \]

\[ \text{...} \]

Solve for coefficients

\[ V = \{ x, y, a, b, q, r \} \]
\[ \deg = 2 \rightarrow x = qy + r, b = ya \]

<table>
<thead>
<tr>
<th>x</th>
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Infer Nonlinear Equations using Equation Solver

- Terms and degrees

\[ V = \{r, y, a\}; \ \text{deg} = 2 \]

\[ T = \{1, r, y, a, ry, ra, ya, r^2, y^2, a^2\} \]

\[ x \quad y \quad a \quad b \quad q \quad r \]
\[ \begin{array}{cccccc}
15 & 2 & 1 & 2 & 0 & 15 \\
15 & 2 & 2 & 4 & 0 & 15 \\
15 & 2 & 1 & 2 & 4 & 7 \\
4 & 1 & 1 & 1 & 0 & 4 \\
4 & 1 & 2 & 2 & 0 & 4 \\
\end{array} \]
Infer Nonlinear Equations using Equation Solver

- Terms and degrees

\[ V = \{ r, y, a \}; \quad \text{deg} = 2 \]

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- Nonlinear equation template

\[ c_1 + c_2 r + c_3 y + c_4 a + c_5 ry + c_6 ra + c_7 ya + c_8 r^2 + c_9 y^2 + c_{10} a^2 = 0 \]
Infer Nonlinear Equations using Equation Solver

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- System of \textit{linear} equations
  \[ \text{trace 1} \rightarrow \{ r = 15, y = 2, a = 1 \} \]
  \[ \text{eq 1} \rightarrow c_1 + 15c_2 + 2c_3 + c_4 + 30c_5 + 15c_6 + 2c_7 + 225c_8 + 4c_9 + c_{10} = 0 \]
  \[ \vdots \]
Infer Nonlinear Equations using Equation Solver

- Terms and degrees
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- System of linear equations
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  \begin{align*}
  \text{trace 1} & \quad \rightarrow \quad \{r = 15, y = 2, a = 1\} \\
  \text{eq 1} & \quad \rightarrow \quad c_1 + 15c_2 + 2c_3 + c_4 + 30c_5 + 15c_6 + 2c_7 + 225c_8 + 4c_9 + c_{10} = 0 \\
  & \quad \vdots \\
\end{align*}
  \]

- Solve for coefficients \(c_i\)
  \[ V = \{x, y, a, b, q, r\}; \quad \text{deg} = 2 \quad \rightarrow \quad x = qy+r, \quad b = ya \]
Static Checking

- **Goal**: prove/refute candidate invariants using program code
- **Approach**: reduce invariant checking to reachability
Static Checking

- **Goal**: prove/refute candidate invariants using program code
- **Approach**: reduce invariant checking to reachability
  - Transform program and invariant into another program consist of a special location \( L' \)

\[
\ldots
\]

\[
[L] \quad //\text{is } x=qy+r \text{ valid?} \quad \implies \quad [L'] \quad //\text{x=qy+r is invalid}
\]

\[
\ldots
\]

\[
\text{if } !(x==qy+r))\{
\]

\[
[L'] 
\quad //\text{x=qy+r is invalid}
\]

\[
\quad \text{abort();}
\]

\[
}\}
\]

\[
//\text{x=qy+r is valid}
\]

\[
\ldots
\]
Static Checking

- **Goal**: prove/refute candidate invariants using program code
- **Approach**: reduce invariant checking to reachability
  - Transform program and invariant into another program consisting of a special location $L'$

```plaintext
... if !(x==qy+r)){
  [L'] //x=qy+r is invalid
  abort();
}
//x=qy+r is valid
...
```

- $L'$ reachable $\implies$ inv is spurious (inputs reaching $L'$ represent cex's)
- $L'$ not reachable (within a time bound) $\implies$ NumInv *accepts* the invariant
Static Checking

**Goal**: prove/refute candidate invariants using program code

**Approach**: reduce invariant checking to reachability

- Transform program and invariant into another program consist of a special location $L'$

```
... if (!(x==qy+r)){
    [L'] //x=qy+r is invalid
    abort();
} //x=qy+r is valid
...
```

- $L'$ reachable $\implies$ inv is spurious (inputs reaching $L'$ represent cex's)
- $L'$ not reachable (within a time bound) $\implies$ NumInv accepts the invariant

**Use the symbolic execution tool** KLEE **to check reachability**

- **Unsound**: KLEE can timeout, but in practice is *very effective* in refuting bad invariants and finding cex's
- Can use other test-input generation tools or verifiers instead of KLEE
Evaluation

Setup

- NumInv is implemented in SAGE/Python (with Z3 backend solver)
- Test machine: 10-core 2.4GHZ CPU, 128GB Ram, Linux OS

Benchmark

- Program Understanding: NLA testsuite, 27 programs with nonlinear invariants
- Complexity Analysis: 19 programs collected from static complexity analysis work
- Program Verification: HOLA benchmark, 46 programs with assertions, compare against PIE
Example: Program Understanding

```c
int cohendiv(int x, int y){
    assert(x>0 && y>0);
    int q=0; int r=x;
    while(r ≥ y){
        int a=1;
        int b=y;
        while[L1](r ≥ 2*b){
            a = 2*a;
            b = 2*b;
        }
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        q=q+a;
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```

What does this program do? What properties hold at **L1** and **L2**?

loop invariants at **L1**:
- \( x = qy + r \)
- \( y ≤ b \)
- \( b ≤ r \)
- \( a ≤ b \)
- \( 2 ≤ a + y \)

postconditions at **L2**:
- \( x = qy + r \)
- \( 1 ≤ q \)
- \( r ≤ y - 1 \)
- \( 0 ≤ r \)
- \( r ≤ x \)

Indicate the exact semantic of integer division and reveal other useful correctness information (e.g., remainder is non-negative)
int cohendiv(int x, int y){
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            a = 2*a;
            b = 2*b;
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}

What does this program do? What properties hold at L1 and L2?

- loop invariants at L1:
  \[ x = qy + r \quad b = ya \]
  \[ y \leq b \quad b \leq r \]
  \[ r \leq x \quad a \leq b \]
  \[ 2 \leq a + y \]

- postconditions at L2:
  \[ x = qy + r \]
  \[ 1 \leq q + r \quad r \leq y - 1 \]
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Example: Program Understanding

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What does this program do? What properties hold at L1 and L2?

- **loop invariants at L1:**
  \[
  x = qy + r \quad b = ya \\
  y ≤ b \quad b ≤ r \\
  r ≤ x \quad a ≤ b \\
  2 ≤ a + y
  \]

- **postconditions at L2:**
  \[
  x = qy + r \\
  1 ≤ q + r \quad r ≤ y - 1 \\
  0 ≤ r \quad r ≤ x
  \]

Indicate the exact semantic of integer division and reveal other useful correctness information (e.g., remainder is non-negative)
### Results: Program Understanding

<table>
<thead>
<tr>
<th>Prog</th>
<th>Locs</th>
<th>Invs</th>
<th>Time (s)</th>
<th>Correct</th>
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### Experiment

- **NLA suite:** 27 programs
- Require nonlinear invariants
- Use documented invariants (loop invariants and postconds) as ground truths
- **Goal:** obtain invariants and compare to ground truths
## Results: Program Understanding

<table>
<thead>
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</table>

### Experiment

- **NLA suite:** 27 programs
- Require nonlinear invariants
- Use documented invariants (loop invariants and postconds) as ground truths
- **Goal:** obtain invariants and compare to ground truths

### Results: NumInv found correct invariants in 23/27 progs

- Most results equiv to or stronger than (imply) ground truths
- Several unexpected and undocumented invariants
- Some invariants reveal “how” program works in details
void triple(int M, int N, int P){
  assert (0 <= M);
  assert (0 <= N);
  assert (0 <= P);
  int i = 0, j = 0, k = 0;
  int t = 0;
  while(i < N){
    j = 0; t++;
    while(j < M){
      j++; k = i; t++;
      while (k < P){
        k++; t++;
      }
      i = k;
    }
    i++;
  }
  [L]
}
Example: Complexity Analysis

```c
void triple(int M, int N, int P){
    assert (0 <= M);
    assert (0 <= N);
    assert (0 <= P);
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    int t = 0;
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            j++; k = i; t++;
            while (k < P){
                k++; t++;
            }
            i = k;
        }
        i++;
    }
    [L]
}
```

Complexity of this program?

- Existing result: \( t = O(N + NM + P) \)
- NumInv found a very unexpected inv:

\[
P^2 Mt + PM^2 t - PMNt - M^2 Nt
- PMt^2 + MNT^2 + PMt - PNT - 2MNT
+ Pt^2 + Mt^2 + Nt^2 - t^3 - Nt + t^2 = 0
\]

Nonlinear invariants can represent disjunctive properties capturing different complexity bounds.
Example: Complexity Analysis

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void triple(int M, int N, int P){
    assert (0 <= M);
    assert (0 <= N);
    assert (0 <= P);
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            while (k < P){
                k++; t++;
            }
            i = k;
        }
        i++;
    }
    [L]
}
```

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\begin{align*}
P^2Mt + PM^2t - PMNt - M^2Nt \\
- PMt^2 + MNt^2 + PMt - PNt - 2MNt \\
+ Pt^2 + Mt^2 + Nt^2 - t^3 - Nt + t^2 = 0
\end{align*}
\]

- Solve for \( t \) yields the most precise, unpublished bound:

\[
\begin{align*}
t &= 0 & \text{when} & & N &= 0, \\
t &= P + M + 1 & \text{when} & & N &\leq P, \\
t &= N - M(P - N) & \text{when} & & N &> P
\end{align*}
\]

- Nonlinear invariants can represent 
  disjunctive properties capturing different complexity bounds
### Results: Complexity Analysis

<table>
<thead>
<tr>
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### Experiment

- **19 progs from static complexity work**
- Obtain postconds representing complexity
- **Goal:** compare against results from prev work
### Results: Complexity Analysis

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### Experiment
- 19 progs from static complexity work
- Obtain postconds representing complexity
- **Goal:** compare against results from prev work

### Results: Obtain equiv (14) or more precise bounds (4) in 18/19 progs
Example: Verification

void f(int u1, int u2) {
    assert(u1 > 0 && u2 > 0);
    int a = 1, b = 1, c = 2, d = 2;
    int x = 3, y = 3;
    int i1 = 0, i2 = 0;
    while (i1 < u1) {
        i1++;
        x = a + c; y = b + d;
        if ((x + y) % 2 == 0) {
            a++; d++; }
        else { a--;}
        i2 = 0;
        while (i2 < u2) {
            i2++; c--; b--;
        }
    }
    [L] //NumInv found:
    //b + 1 = c, a + 1 = d,
    //a + b <= 2, 2 <= a
    assert(a + c == b + d);
}

void g(int n, int u1) {
    assert(u1 > 0);
    int x = 0;
    int m = 0;
    while (x < n) {
        if (u1) {
            m = x;
        }
        x = x + 1;
    }
    [L] //NumInv found:
    //m^2 = nx - m - x, mn = x^2 - x
    //m <= x, x <= m + 1, n <= x
    if (n > 0) {
        assert(0 <= m && m < n);
    }
}
Results: Verification

Experiment

- HOLA benchmark: 46 programs
- Various assertions (mostly postconds)
- **Goal:**
  - Obtain and compare invariants: if match or imply assertions, then assertions hold
  - Also compare with existing tool PIE

Results: Found equiv (23) or stronger (13) invariants in 36/46 programs

Time: mean 30s, median 13s

Nonlinear invariants can prove many nontrivial and unsupported properties.
Results: Verification

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Conclusion

NumInv

- Use CEGIR for *numerical* invariant generation
  - Dynamic Inference: use DIG to compute nonlinear invariants
  - Static Checking: use KLEE to check candidate invariants and obtain cex’s

- *Unsound*, but experience shows practical and effective in removing invalid results and can handle complex invariants
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  - Static Checking: use KLEE to check candidate invariants and obtain cex’s
- *Unsound*, but experience shows practical and effective in removing invalid results and can handle complex invariants

Results

- Discover necessary nonlinear invariants to understand programs
- Find useful invariants capturing nontrivial runtime complexity
- Compete well with existing work
- General polynomial invariants (e.g., nonlinear properties) can *surprisingly* represent/prove many nontrivial, complex, and *unsupported* properties

https://bitbucket.org/nguyenthanhvuh/dig2/