Computer Science & Engineering 423/823
Design and Analysis of Algorithms

Lecture 07 — Maximum Flow (Chapter 26)

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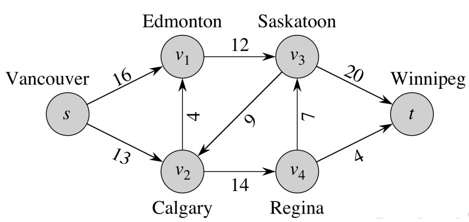
(Adapted from Vinodchandran N. Variyam)

Introduction

- Can use a directed graph as a *flow network* to model:
  - Data through communication networks, water/oil/gas through pipes, assembly lines, etc.
- A *flow network* is a directed graph with two special vertices: *source* s that produces flow and *sink* t that takes in flow
- Each directed edge is a conduit with a certain capacity (e.g. 200 gallons/hour)
- Vertices are conduit junctions
- Except for s and t, flow must be conserved: The flow into a vertex must match the flow out
- *Maximum flow problem:* Given a flow network, determine the maximum amount of flow that can get from s to t
- Other application: Bipartite matching

Flow Networks

- A *flow network* $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative *capacity* $c(u, v) \geq 0$
- If $(u, v) \notin E$, $c(u, v) = 0$
- Assume that every vertex in V lies on some path from the *source vertex* $s \in V$ to the *sink vertex* $t \in V$



Flows

- A *flow* in graph G is a function $f : V \times V \to \mathbb{R}$ that satisfies:
  1. **Capacity constraint:** For all $u, v \in V$, $0 \leq f(u, v) \leq c(u, v)$ (flow should be nonnegative and not exceed capacity)
  2. **Flow conservation:** For all $u \in V \setminus \{s, t\}$,

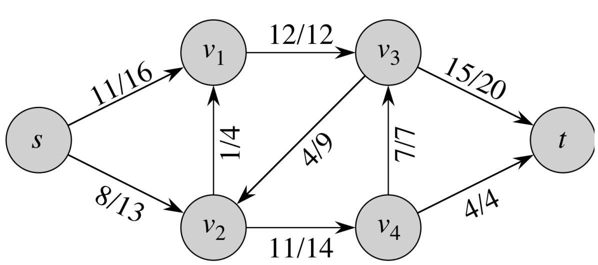
$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)$$

  (flow entering a vertex = flow leaving)
- The *value* of a flow is the net flow out of s ($=$ net flow into t):

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

- *Maximum flow problem:* given graph and capacities, find a flow of maximum value

Flow Example

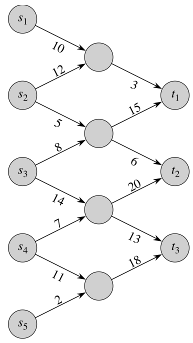


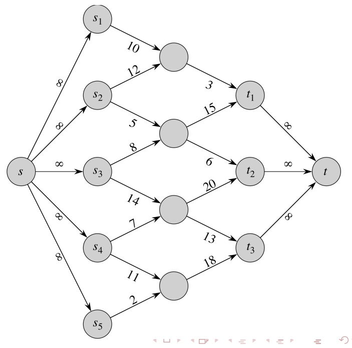
What is the value of this flow?

Multiple Sources and Sinks

- Might have cases where there are multiple sources and/or sinks; e.g. if there are multiple factories producing products and/or multiple warehouses to ship to
- Can easily accommodate graphs with multiple sources $s_1, \ldots, s_k$ and multiple sinks $t_1, \ldots, t_\ell$
- Add to G a *supersource* s with an edge (s, s_i) for $i \in \{1, \ldots, k\}$ and a *supersink* t with an edge (t_j, t) for $j \in \{1, \ldots, \ell\}$
- Each new edge has a capacity of ∞

Multiple Sources and Sinks (2)





Ford-Fulkerson Method

- A *method* (rather than specific algorithm) for solving max flow
- Multiple ways of implementing, with varying running times
- Core concepts:
  1. *Residual network:* A network G_f, which is G with capacities reduced based on the amount of flow f already going through it
  2. *Augmenting path:* A simple path from s to t in residual network G_f
     - ⇒ If such a path exists, then can push more flow through network
  3. *Cut:* A partition of V into S and T where $s \in S$ and $t \in T$; can measure *net flow* and *capacity* crossing a cut
- Method repeatedly finds an augmenting path in residual network, adds in flow along the path, then updates residual network

Ford-Fulkerson-Method(G, s, t)

Initialize flow f to 0 ;
while *there exists augmenting path* p *in residual network* G_f **do**
    augment flow f along p ;
end
return f ;

Residual Networks

- Given flow network G with capacities c and flow f, *residual network* G_f consists of edges with capacities showing how one can change flow in G
- Define *residual capacity* of an edge as

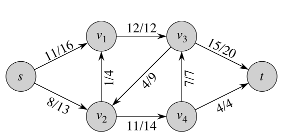
$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \\ 0 & \text{otherwise} \end{cases}$$

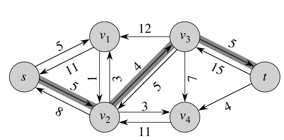
- E.g. if $c(u,v) = 16$ and $f(u,v) = 11$, then $c_f(u,v) = 5$ and $c_f(v,u) = 11$
- Then can define $G_f = (V, E_f)$ as

$$E_f = \{(u,v) \in V \times V : c_f(u,v) > 0\}$$

- So G_f will have some edges not in G, and vice-versa

Residual Networks (2)





Flow Augmentation

- G_f is like a flow network (except that it can have an edge and its reversal); so we can find a flow within it
- If f is a flow in G and f' is a flow in G_f, can define the *augmentation* of f by f' as

$$(f \uparrow f')(u,v) = \begin{cases} f(u,v) + f'(u,v) - f'(v,u) & \text{if } (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$$

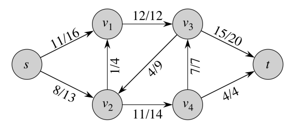
- **Lemma:** $f \uparrow f'$ is a flow in G with value $|f \uparrow f'| = |f| + |f'|$
- **Proof:** Not difficult to show that $f \uparrow f'$ satisfies capacity constraint and and flow conservation; then show that $|f \uparrow f'| = |f| + |f'|$ (pp. 718–719)
- Result: If we can find a flow f' in G_f, we can increase flow in G

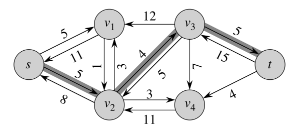
Augmenting Path

- By definition of residual network, an edge $(u, v) \in E_f$ with $c_f(u, v) > 0$ can handle additional flow
- Since edges in E_f all have positive residual capacity, it follows that if there is a simple path p from s to t in G_f, then we can increase flow along each edge in p, thus increasing total flow
- We call p an *augmenting path*
- The amount of flow we can put on p is p's residual capacity:

$$c_f(p) = \min\{c_f(u, v) : (u, v) \text{ is on } p\}$$

Augmenting Path (2)





p is shaded; what is $c_f(p)$?

Augmenting Path (3)

- **Lemma:** Let $G = (V, E)$ be a flow network, f be a flow in G, and p be an augmenting path in G_f. Define $f_p : V \times V \to \mathbb{R}$ as

$$f_p(u, v) = \begin{cases} c_f(p) & \text{if } (u, v) \in p \\ 0 & \text{otherwise} \end{cases}$$

  Then f_p is a flow in G_f with value $|f_p| = c_f(p) > 0$
- **Corollary:** Let G, f, p, and f_p be as above. Then $f \uparrow f_p$ is a flow in G with value $|f \uparrow f_p| = |f| + |f_p| > |f|$
- Thus, every augmenting path increases flow in G
- When do we stop? Will we have a maximum flow if there is no augmenting path?

Max-Flow Min-Cut Theorem

- Used to prove that once we run out of augmenting paths, we have a maximum flow
- A *cut* (S, T) of a flow network $G = (V, E)$ is a partition of V into $S \subseteq V$ and $T = V \setminus S$ such that $s \in S$ and $t \in T$
- *Net flow* across the cut (S, T) is

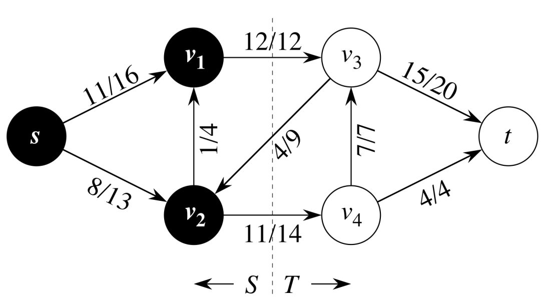
$$f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u)$$

- *Capacity* of cut (S, T) is

$$c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v)$$

- A *minimum cut* is one whose capacity is smallest over all cuts

Max-Flow Min-Cut Theorem (2)



What are $f(S, T)$ and $c(S, T)$?

Max-Flow Min-Cut Theorem (3)

- **Lemma:** For any flow f, the value of f is the same as the net flow across any cut; i.e. $f(S, T) = |f|$ for all cuts (S, T)
- **Corollary:** The value of any flow f in G is upperbounded by the capacity of **any** cut of G
- **Proof:**

$$\begin{aligned} |f| &= f(S, T) \\ &= \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u) \\ &\leq \sum_{u \in S} \sum_{v \in T} f(u, v) \\ &\leq \sum_{u \in S} \sum_{v \in T} c(u, v) \\ &= c(S, T) \end{aligned}$$

Max-Flow Min-Cut Theorem (4)

- **Max-Flow Min-Cut Theorem:** If f is a flow in flow network G, then these statements are equivalent:
  1. f is a maximum flow in G
  2. G_f has no augmenting paths
  3. $|f| = c(S,T)$ for some (i.e. minimum) cut (S,T) of G
- **Proof:** Show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1)
- **(1) $\Rightarrow$ (2):** If G_f has augmenting path p, then $f_p > 0$ and $|f \uparrow f_p| = |f| + |f_p| > |f| \Rightarrow$ contradiction that f is a max flow

Max-Flow Min-Cut Theorem (5)

- **(2) $\Rightarrow$ (3):** Assume G_f has no path from s to t and define $S = \{u \in V : s \leadsto u \text{ in } G_f\}$ and $T = V \setminus S$
  - (S,T) is a cut since it partitions V, $s \in S$ and $t \in T$
  - Consider $u \in S$ and $v \in T$:
    - If $(u,v) \in E$, then $f(u,v) = c(u,v)$ since otherwise $c_f(u,v) > 0 \Rightarrow (u,v) \in E_f \Rightarrow v \in S$
    - If $(v,u) \in E$, then $f(v,u) = 0$ since otherwise we'd have $c_f(u,v) = f(v,u) > 0 \Rightarrow (u,v) \in E_f \Rightarrow v \in S$
    - If $(u,v) \notin E$ and $(v,u) \notin E$, then $f(u,v) = f(v,u) = 0$
  - Thus (by applying the Lemma as well)

$$\begin{aligned} |f| = f(S,T) &= \sum_{u\in S}\sum_{v\in T} f(u,v) - \sum_{v\in T}\sum_{u\in S} f(v,u) \\ &= \sum_{u\in S}\sum_{v\in T} c(u,v) - \sum_{v\in T}\sum_{u\in S} 0 = c(S,T) \end{aligned}$$

Max-Flow Min-Cut Theorem (6)

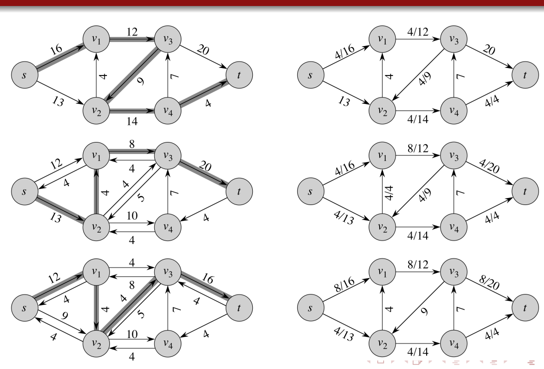
- **(3) $\Rightarrow$ (1):**
  - Corollary says that $|f| \le c(S',T')$ for all cuts (S',T')
  - We've established that $|f| = c(S,T)$
    - $\Rightarrow$ $|f|$ can't be any larger
    - $\Rightarrow$ f is a maximum flow

Ford-Fulkerson(G,s,t)

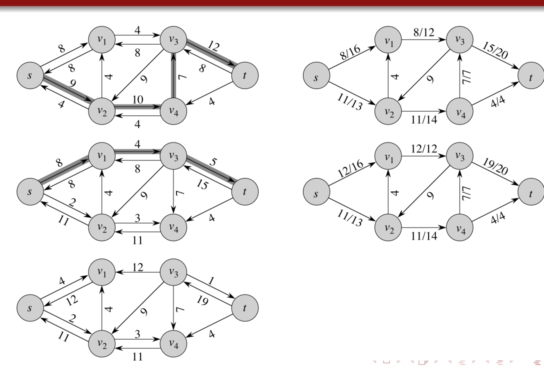
```
for each edge (u,v) ∈ E do
    f(u,v) = 0 ;
end
while there exists path p from s to t in G_f do
    c_f(p) = min{c_f(u,v) : (u,v) is in p} ;
    for each edge (u,v) ∈ p do
        if (u,v) ∈ E then
            f(u,v) = f(u,v) + c_f(p) ;
        else
            f(v,u) = f(v,u) − c_f(p) ;

    end
end
```


Ford-Fulkerson Example



Ford-Fulkerson Example (2)

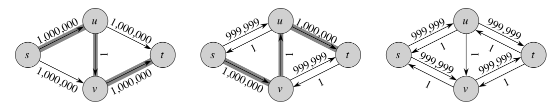


Analysis of Ford-Fulkerson

- Assume all of G's capacities are integers
  - If not, but values still rational, can scale them
  - If values irrational, might not converge ⌢̈
- If we choose augmenting path arbitrarily, then $|f|$ increases by at least one unit per iteration $\Rightarrow$ number of iterations is $\leq |f^*| =$ value of max flow
- $|E_f| \leq 2|E|$
- Every vertex is on a path from s to $t \Rightarrow |V| = O(|E|)$
- ⇒ Finding augmenting path via BFS or DFS takes time $O(|E|)$, as do initialization and each augmentation step
- Total time complexity: $O(|E||f^*|)$
- Not polynomial in size of input! (What is size of input?)

Example of Large $|f^*|$

Arbitrary choice of augmenting path can result in small increase in $|f|$ each step



Takes 2×10^6 augmentations

Edmonds-Karp Algorithm

- Uses Ford-Fulkerson Method
- Rather than arbitrary choice of augmenting path p from s to t in G_f, choose one that is shortest in terms of number of edges
  - How can we easily do this?
- Will show time complexity of $O(|V||E|^2)$, independent of $|f^*|$
- Proof based on $\delta_f(u,v)$, which is length of shortest path from u to v in G_f, in terms of number of edges
- **Lemma:** When running Edmonds-Karp on G, for all vertices $v \in V \setminus \{s,t\}$, shortest path distance $\delta_f(u,v)$ in G_f increases monotonically with each flow augmentation

Edmonds-Karp Algorithm (2)

- **Theorem:** When running Edmonds-Karp on G, the total number of flow augmentations is $O(|V||E|)$
- **Proof:** Call an edge (u,v) *critical* on augmenting path p if $c_f(p) = c_f(u,v)$
- When (u,v) is critical for the first time, $\delta_f(s,v) = \delta_f(s,u) + 1$
- At the same time, (u,v) disappears from residual network and does not reappear until its flow decreases, which only happens when (v,u) appears on an augmenting path, at which time

$$\begin{aligned}\delta_{f'}(s,u) &= \delta_{f'}(s,v) + 1 \\ &\geq \delta_f(s,v) + 1 \text{ (from Lemma)} \\ &= \delta_f(s,u) + 2\end{aligned}$$

- Thus, from the time (u,v) becomes critical to the next time it does, u's distance from s increases by at least 2

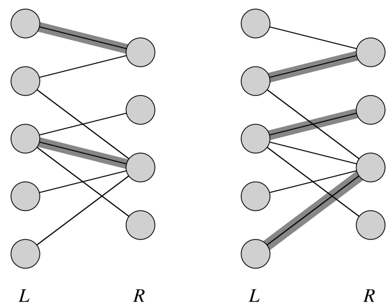
Edmonds-Karp Algorithm (3)

- Since u's distance from s is at most $|V| - 2$ (because $u \neq t$) and at least 0, edge (u,v) can be critical at most $|V|/2$ times
- There are at most $2|E|$ edges that can be critical in a residual network
- Every augmentation step has at least one critical edge
- ⇒ Number of augmentation steps is $O(|V||E|)$, instead of $O(|f^*|)$ in previous algorithm
- ⇒ Edmonds-Karp time complexity is $O(|V||E|^2)$

Maximum Bipartite Matching

- In an undirected graph $G = (V,E)$, a *matching* is a subset of edges $M \subseteq E$ such that for all $v \in V$, at most one edge from M is incident on v
- If an edge from M is incident on v, v is *matched*, otherwise *unmatched*
- **Problem:** Find a matching of maximum cardinality
- **Special case:** G is *bipartite*, meaning V partitioned into disjoint sets L and R and all edges of E go between L and R
- Applications: Matching machines to tasks, arranging marriages between interested parties, etc.

Bipartite Matching Example

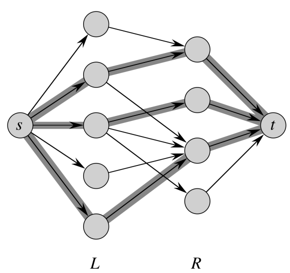


$|M| = 2$ $\qquad$ $|M| = 3$ (maximum)

Casting Bipartite Matching as Max Flow

- Can cast bipartite matching problem as max flow
- Given bipartite graph $G = (V, E)$, define *corresponding flow network* $G' = (V', E')$:
$$V' = V \cup \{s, t\}$$
$$E' = \{(s, u) : u \in L\} \cup \{(u, v) : (u, v) \in E\} \cup \{(v, t) : v \in R\}$$
- $c(u, v) = 1$ for all $(u, v) \in E'$

Casting Bipartite Matching as Max Flow (2)



Value of flow across cut $(L \cup \{s\}, R \cup \{t\})$ equals $|M|$

Casting Bipartite Matching as Max Flow (3)

- **Lemma:** Let $G = (V, E)$ be a bipartite graph with V paritioned into L and R and let $G' = (V', E')$ be its corresponding flow network. If M is a matching in G, then there is an *integer-valued flow* f in G' with value $|f| = |M|$. Conversely, if there is an integer-valued flow f in G', then there is a matching M in G with cardinality $|M| = |f|$.
- **Proof:** $\Rightarrow$ If $(u, v) \in M$, set $f(s, u) = f(u, v) = f(v, t) = 1$
  - Set flow of all other edges to 0
  - Flow satisfies capacity constraint and flow conservation
  - Flow across cut $(L \cup \{s\}, R \cup \{t\})$ is $|M|$
  - $\Leftarrow$ Let f be integer-valued flow in G', and set
$$M = \{(u, v) : u \in L, v \in R, f(u, v) > 0\}$$
  - Any flow into u must be exactly 1 in and exactly 1 out on one edge
  - Similar argument for $v \in R$, so M is a matching with $|M| = |f|$

Casting Bipartite Matching as Max Flow (4)

- **Theorem:** If all edges in a flow network have integral capacities, then the Ford-Fulkerson method returns a flow with value that is an integer, and for all $(u, v) \in V$, $f(u, v)$ is an integer
- Since the corresponding flow network for bipartite matching uses all integer capacities, can use Ford-Fulkerson to solve matching problem
- Any matching has cardinality $O(|V|)$, so the corresponding flow network has a maximum flow with value $|f^*| = O(|V|)$, so time complexity of matching is $O(|V||E|)$