

Introduction to NP-Complete Problems

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Definitions
The 4-Step Proof
Example 1: Vertex Cover
Example 2: Jogger

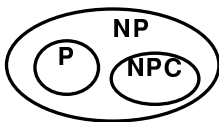
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\mathcal{P} , \mathcal{NP} and \mathcal{NP} -Complete

Given a problem, it belongs to \mathcal{P} , \mathcal{NP} or \mathcal{NP} -Complete classes, if:

- ▶ \mathcal{NP} : verifiable in polynomial time.
- ▶ \mathcal{P} : decidable in polynomial time.
- ▶ \mathcal{NP} -Complete: all problems in \mathcal{NP} can be reduced to it in polynomial time.

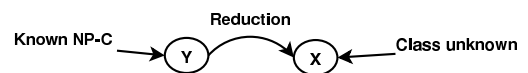


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The 4-Step Proof

Given a problem X , prove it is in \mathcal{NP} -Complete.

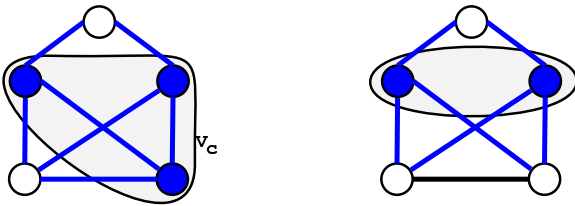
1. Prove X is in \mathcal{NP} .
2. Select problem Y that is known to be in \mathcal{NP} -Complete.
3. Define a polynomial time reduction from Y to X .
4. Prove that given an instance of Y , Y has a solution *iff* X has a solution.



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Vertex Cover

- ▶ A *vertex cover* of a graph $G = (V, E)$ is a $V_C \subseteq V$ such that every $(a, b) \in E$ is incident to at least a $u \in V_C$.



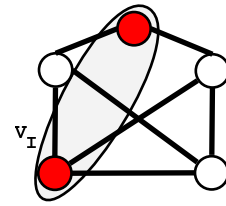
→ Vertices in V_C 'cover' all the edges of G .

- ▶ The VERTEX COVER (VC) decision problem: Does G have a vertex cover of size k ?

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Independent Set

- ▶ An *independent set* of a graph $G = (V, E)$ is a $V_I \subseteq V$ such that no two vertices in V_I share an edge.



→ $u, v \in V_I$ cannot be neighbors.

- ▶ The INDEPENDENT SET (IS) decision problem: Does G have an independent set of size k ?

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Prove VERTEX COVER is \mathcal{NP} -complete

Given that the INDEPENDENT SET (IS) decision problem is \mathcal{NP} -complete, prove that VERTEX COVER (VC) is \mathcal{NP} -complete.

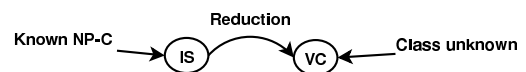
Solution:

1. Prove VERTEX COVER is in \mathcal{NP} .
 - ▶ Given V_C , vertex cover of $G = (V, E)$, $|V_C| = k$
 - ▶ We can check in $O(|E| + |V|)$ that V_C is a vertex cover for G . How?
 - ▶ For each vertex $u \in V_C$, remove all incident edges.
 - ▶ Check if all edges were removed from G .
 - ▶ Thus, VERTEX COVER $\in \mathcal{NP}$

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Prove VERTEX COVER is \mathcal{NP} -complete (2)

2. Select a known \mathcal{NP} -complete problem.
 - ▶ INDEPENDENT SET (IS) is a known \mathcal{NP} -complete problem.
 - ▶ Use IS to prove that VC is \mathcal{NP} -complete.



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Prove VERTEX COVER is \mathcal{NP} -complete (3)

3. Define a polynomial-time reduction from IS to VC:

- ▶ Given a general instance of IS: $G'=(V', E'), k'$
- ▶ Construct a specific instance of VC: $G=(V, E), k$
 - ▶ $V=V'$
 - ▶ $E=E'$
 - ▶ $(G=G')$
 - ▶ $k=|V'| - k'$
- ▶ This transformation is polynomial:
 - ▶ Constant time to construct $G=(V, E)$
 - ▶ $O(|V|)$ time to count the number of vertices
- ▶ Prove that there is a V_I ($|V_I| = k'$) for G' iff there is an V_C ($|V_C| = k$) for G .

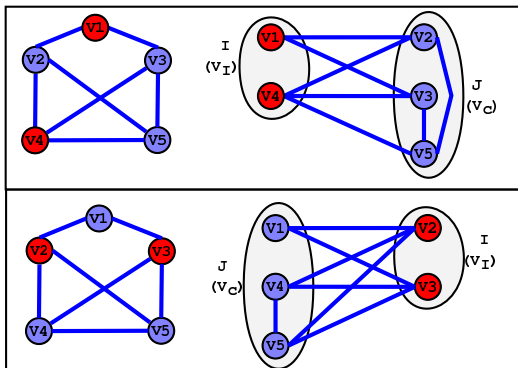
Prove VERTEX COVER is \mathcal{NP} -complete (4)

Prove G' has an independent set V_I of size k' iff VC has a vertex cover V_C of size k .

- ▶ Consider two sets I and J s.t. $I \cap J = \emptyset$ and $I \cup J = V = V'$
- ▶ Given any edge (u, v) , one of the following four cases holds:
 1. $u, v \in I$
 2. $u \in I$ and $v \in J$
 3. $u \in J$ and $v \in I$
 4. $u, v \in J$
- ▶ Assume that I is an independent set of G' then:
 - ▶ Case 1 cannot be; (vertices in I cannot be adjacent)
 - ▶ In cases 2 and 3, (u, v) has *exactly one* endpoint in J .
 - ▶ In case 4, (u, v) has *both* endpoints in J .
 - ▶ In cases 2, 3 and 4, (u, v) has *at least one* endpoint J .
 - ▶ Thus, vertices in J cover all edges of G' .
 - ▶ Also: $|J| = |V| - |I|$ since $I \cap J = \emptyset$ and $I \cup J = V = V'$
 - ▶ Thus, if I is an independent set of G' , then J is a vertex cover of $G' (= G)$.

Similarly, we can prove that if J is a vertex cover for G' , then I is an independent set for G' . □

Prove VERTEX COVER is \mathcal{NP} -complete (5)



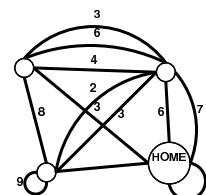
Jogger Problem

Given a weighted, undirected graph G with:

- ▶ loops, multiple edges, and only positive weights,
- ▶ a special node v called *home*,
- ▶ and given an integer $i \geq 0$.

Is there a route for a jogger J that:

- ▶ starts from home,
- ▶ travels a distance i , and
- ▶ returns home
- ▶ without repeating an edge (nodes can be repeated)?



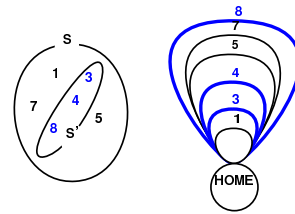
Jogger is \mathcal{NP} -complete

1. Jogger is in \mathcal{NP} : Given a path P , we can check in $O(|P|)$ whether or not the sum of all edge weights is equal to i .
2. Consider the Subset Sum (SS) problem¹, which is a known \mathcal{NP} -complete problem.
Given a set S of positive integers, is there a subset $S' \subseteq S$ such that sum of the elements of S' is t .
Example: $S = \{1, 3, 4, 5, 7, 8\}$, find $S' \subseteq S$ such that sum of the elements in S' is 15.
3. Reduce Subset Sum to the Jogger.

¹One type of knapsack problem.

Jogger is \mathcal{NP} -complete

- ▶ Given an instance of SS: $S = \{a_1, a_2, \dots, a_n\}$, construct a graph G as follows:
 - ▶ G has a unique node, v , which is the home.
 - ▶ For each $a_i \in S$, add a self-loop to v of weight a_i .
 - ▶ Let i (of the Jogger) = t (of the Sum Set).
- ▶ This construction is obviously linear in the number of elements in S .



Jogger is \mathcal{NP} -complete (2)

4. SS has a solution *iff* Jogger has a solution.
 - ▶ G contains a path starting from home, never repeating an edge, and returning back home with a total distance exactly i *iff* S has a subset S' with sum of elements of S' equal to t .
 - ▶ If $S' \subseteq S$ is a solution to SS, then the Jogger has a path of length $i = t$ by taking the edges (loops) corresponding to the elements in S' .
 - ▶ If there a path P is a solution to Jogger, then the subset of S with elements corresponding to the edges in P is a subset with sum $i = t$ and thus is a solution to SS. \square

