ROUNDING ERRORS LAB

Imagine you are traveling in Italy, and you are trying to convert $27.00 into Euros. You go to the bank teller, who gives you €20.19. Your friend is with you, and she is converting $2,700.00. You predict that she will get

100 * €20.19 = €2019.00,

but to your dismay, she gets €2019.11. Where did the extra €0.11 come from?[1]

1. OBJECTIVE

In this lab, we will explore a very common error in programming: rounding errors. We will discuss how different representations of numbers in your computer can change the values that you get when doing calculations, and how these differences can have serious consequences in your applications.

2. INTRODUCTION

Floating point numbers

In most programming languages, including Java, C++, and C, there are multiple data types that are used to represent numbers. The most common types are `int`, `double`, and `float`. These types can be `signed` or `unsigned`. We will briefly introduce these options below.

- `int` is used to represent integer values without any decimals.
- Decimal numbers can be represented in multiple ways:
  - fixed-point or floating-point, and
  - single or double precision.
- `unsigned` means only positive numbers can be represented
- `signed` means both positive and negative numbers can be represented

[1] A conversion rate of 0.74782 euros to a dollar was used. You calculate $27*0.74782 and realize it comes to €20.1911, but since there are no €0.001 in euros, your amount was rounded down to €20.19. This is one example of rounding errors that can cause serious problems in computer software if they go unchecked.

What are fixed point and floating point numbers?

Fixed vs. floating point refers to the position of the decimal point in the stored value. Fixed-point means the decimal point is always in the same position, i.e. we always have the same number of digits (in binary) before and after. For example, in base 10, we might decide to always have 2 digits before the decimal and 2 digits after, so numbers would look like 10.00 or 00.01.

Floating point means the decimal point may be in any position of the number. The computer stores floating point numbers the way we write scientific notation in base 10: there is a significand (the number part of scientific notation), and exponent (the power of 10 part of scientific notation), as well as sign bits for each (which we will discuss later). For example, we could have $1.09x10^3$ or $3.45x10^{-5}$. On a computer, however, the numbers are represented base 2 (binary) intead of base 10. Notice that the decimal point floats to different places in the number: 1090 int the first case and 0.0000345 in the second.

Why is floating point useful?

Floating point numbers allow you to represent a huge range of values that `int`s and fixed point `doubles` do not. Imagine you have 4 digits with which to represent all numbers. With integers, you can represent numbers from 0001 to 9999. With fixed point, you might get the range 00.01 to 99.99. But say you use the first 3 digits for the significand, and the 4th for the exponent. Then you can represent $0.01x10^0$ to $9.99x10^9$, which is a much wider range!

Computer Representation

In computer memory, everything is represented in binary (base 2) instead of decimal (base 10). Each digit is either 0 or 1, and each place is a power of 2 rather than 10. So, for example, the floating point representation of $2.375x10^0$ is actually $1.0011x2^1$ in binary.

Aside: Converting to binary

When you declare an `int` or a `double` in a programming language, the computer stores it for you in memory, automatically taking care of the details of its representation as an integer or floating- or fixed-point decimal. However, it is useful to understand how numbers are written in binary to avoid the types of errors that we will discuss later on.

Here is a quick tutorial of how to convert an `int` to its binary representation. Just follow these steps:

1) Take the number and divide it by 2
2) If the remainder is 1, the least significant bit (2^0) is 1. If it divides evenly, the bit is 0.
3) Repeat steps 1 and 2 on the result of the division. Each bit you add will be for the next power of 2 (build the binary number from right to left). Stop when the result is 0.

Take the example 13:

13/2 = 6 remainder 1, so our 2^0 bit is 1.
6/2 = 3, so our 2^1 bit is 0. We now have 01.
3/2 = 1 remainder 1, so our 2^2 bit is 1. We now have 101.
1/2 = 0 remainder 1, so our 2^3 bit is 1. We now have 1101.

Since our result is now 0, we stop and our final number is 1101.
We can convert back to base 10 by multiplying the bit value in each position by 2 to the power of the position number.
In this example, 1101 is $2^3*1 + 2^2*1 + 2^1*0 + 2^0*1 = 8+4+1 = 13$. Success!

Every integer value can be represented as an exact binary number (in the computer, every integer value in the range stated above). Consider why this is true - if we divide by 2 each time, eventually we will get to 0 with remainder 1 (If you don't believe this, try it out on a few numbers!). This will always be our most significant bit.

Note: To get the binary representation of a negative integer, we want to use the "two's complement." This means flipping all of the bits and adding 1 to the result. To show that it is negative, we add a sign bit of 1 before the binary. For example, if we want -9, we take 9=1001 in binary, flip the bits to get 0110, add 1 to get 0111, and add a 1 as the sign bit for a final 10111. To be able to understand this binary, we have to know if the binary is signed so we look for a sign bit. Then we can do the reverse of what we just did instead of reading this as the unsigned 23.

Decimal values are trickier, since many decimals cannot be represented exactly in binary. For the integer part of the decimal value, our conversion is the same as above. For the decimal part, however, we go through a similar process, but multiplying by 2 instead of dividing, and building the number left to right. We add bits one at a time after the decimal point, starting with the 2^{-1} bit, then 2^{-2}, 2^{-3}, etc.).

1) Take the decimal number and multiply it by 2.
2) If you get a number <1, your next bit is 0. Return to step 1.
3) If you get a number >1, your next bit is a 1. Subtract 1 from the number, and return to step 1
4) If you get exactly 1, your next bit is 1 and the algorithm is complete: you have found the exact representation.

You might be able to see why this can be an issue - while division always gets down to the base case of 0 remainder 1, with multiplication there is no obvious stopping point unless you reach 1.

Consider trying to convert 0.2 to binary:

0.2*2=0.4, so our first bit is 0, giving .0.
0.4*2=0.8, so we now have .00.
0.8*2=1.6, so our next bit is 1, giving .001.
We now take 1.6-1=0.6, and continue with 0.6*2=1.2. This gives us .0011, and we take 1.2-1=0.2 as our next number.

We are now back with 0.2, which is where we started. If we continue this process we will repeat this cycle infinitely, never reaching 1 exactly. Our binary number is 0.0011001100110011..., but it has no exact representation.

This will be important to remember as we discuss floating point numbers and rounding errors in this lab!

This is where single vs. double precision floats come into play. Single precision means that the number uses one “word” of memory, which in most operating systems in 32 bits, to represent the number. A double precision uses two words, i.e. 64 bits, and therefore can be more precise.

When these values are signed, they have to split their available range between positive and negative numbers, which means that they cannot reach as high a positive number as an unsigned int. This is because one of these 32 (or 64 for `doubles`) bits is reserved to mark whether the number is positive (0) or negative (1).

Ranges of Numeric Types

Type	**Min**	**Max**
Unsigned int	0	4,294,967,295
Signed int	-2,147,483,648	2,147,483,647
Signed double	-1.79769e+308	1.79769e+308

Source: defined as constants in C++ in “limits.h” for `ints` and <cfloat> for `doubles`

For a 32 bit unsigned float, there are 23 bits for the significand (in the format X.XXXXX...), a sign bit for the exponent that is 0 for positive and 1 for negative, and 8 digits for the exponent value.

If it is a signed float, there are 2 sign bits (one for the significand, one for the exponent) and one less bit used for the significand. We will continue to use the base 10 analogy in this example, since it is more familiar for many students.

For more information, check out this series of YouTube videos about floating points and their representations:
Floating Point Background: http://www.youtube.com/watch?v=svFJXukm2uE
Floating Point Example: http://www.youtube.com/watch?v=t-8fMtUNX1A

So what's the problem?

There are several problems. We describe them below, demonstrating what can go wrong when we are using 4 decimal digits to represent a floating point number. Keep in mind that on a computer, binary bits are used instead of decimal digits, and there are many more bits than 4, but the same problems still apply.

Imprecision: Say you wanted to represent 93,425. We can write 9.34×10^4 (93,400) or 93.5×10^4 (93,500), but not 93,425 exactly. What if we tried to do $9.34 \times 10^4 + 2.50 \times 10^1$ to get 93,425? When we add these numbers together, we can still only represent 3 significant digits, and we again get 9.34×10^4. This can cause significant errors in calculations. If you tried to add 2.50×10^1 to 9.34×10^4 40 times, you should get 9.44×10^4, but instead you would just get 9.34×10^4 using this logic.

Rounding Errors: Suppose you wish to represent the number 1/7 in decimal. This number goes on infinitely repeating the sequence 0.142857. How would you represent this in the above representation? 1.43×10^{-1} is as close as you can get. Even with 10, 20, or 100 digits, you would have to have some rounding to represent an infinite number in a finite space. If you have a lot of digits, your rounding error might seem insignificant. But consider what happens if you add up these rounded numbers repeatedly for a long period of time. If you round 1/7 to 1.42×10^{-1}, and add up this representation of 700 times, you would expect to get 100, or 1.00×10^2. However, you instead get 9.94×10^2.

Base 2 confusion: In binary, rounding errors occur in fractions that cannot be represented as powers of 2 (rather than base 10 like the examples above). In fact, there is no way to represent 1/10 exactly in binary, even as a floating point with 23 digits. We can get a close approximation, but this is often not good enough - as you will see in the later example of the Patriot Missile disaster, these rounding errors can sometimes have huge consequences. A power of 2 like ⅛ (i.e. 2^{-3}), on the other hand, is exactly 0.001 in binary, and there is no rounding needed.

Why should we care?

Relatively small imprecisions and rounding errors like the ones described above can have huge impacts. In this lab, you will look at a few real world examples of the problems these rounding errors can cause. Knowing how these rounding errors can occur and being conscious of them will help you be a better and more precise programmer.

3. EXERCISES

Note to instructors: Some of these examples rely on 32-bit representation of floats. If this is an issue, there is a flag "-mpc32" that can be used which will force the gcc compiler to use 32-bit floating points. For a list of flags please see http://gcc.gnu.org/onlinedocs/gcc/i386-and-x86_002d64-Options.html

3.1. TRY IT YOURSELF!

Now that we've discussed the difficulty of representing 1/7 in base 10, try representing 1/7 in base 2, and add up this representation 700 times.

```
#include <iostream>

using namespace std;

float addFraction();

int main() {
   cout << addFraction() << endl;
   return 0;
}

float addFraction(){
   float fraction = (float)1/7;
   // Your code here










}
```

1) Run this program. What is the total after adding the fraction 1/7, 700 times?

2) Now change the fraction from 1/7 to 1/6 and add this up 600 times, what is your answer now?

3) Which one of these fractions was rounded up? Which one was rounded down?

4) Change the fraction to 1/8 and add this 800 times. What is the answer? Why? (Hint: Think of the binary representation for powers of 2.)[2]

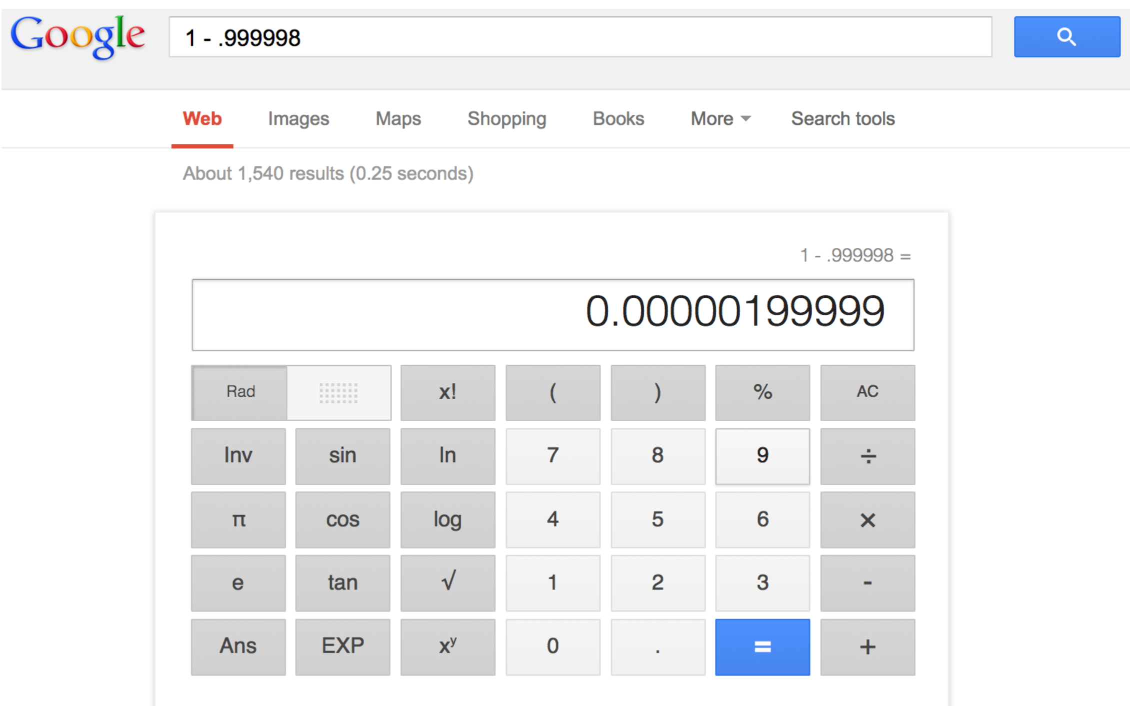
[2] A good source to convert between base 10 to base 2 is http://www.binaryconvert.com/

3.2. GOOGLE CALCULATOR

You are in the library and need to do some calculations for your physics homework. However, you just realized that you forgot your calculator at home and decide to use Google calculator for your answers.

Here is the calculation you need to make. You have two objects. Object one traveled 1.0 meter and object two traveled .999998 meters. You want to find the difference between the two, so you go to google.com and search for: *1.0 - 0.999998*. To your dismay, Google calculator tells you that *0.00000199999* is your answer! You know that this is wrong, since the answer is clearly 0.000002.

What could have possibly gone wrong?



[3]

[3] Source: Google and the Google logo are registered trademarks of Google Inc., used with permission.

3.2.1 Representation of numbers (Google v. Bing)

Now that you found this error, you decide to try out a few more calculations in search of an answer.

First, go to Google[4] to see this error with your own eyes, search for: *1.0 - 0.999998*
What do you get?

You remember that a friend told you that Bing is better at math, so you try the same search, *1.0 - 0.999998*. What do you get now?

Bing and Google show you different answers for the same calculation, but only one of the two calculations is right. What could have Bing done differently from Google in this situation?

While the actual implementation for these numbers is unknown to the public, the most likely explanation for these errors lies in the underlying binary representation of numbers by each of these search engines.

[4] To get to the Google calculator, simply go to google.com and type the equation into the search bar. If this does not bring up the Google calculator, try adding "=" (without the quotes) after the equation.

3.2.2 Write your own calculator: The difference between `int`, `float`, and `double`

As previously discussed, there are multiple data types that are used when representing numbers. The three main types are `float`, `double`, and `int`. In this section, you'll explore which type works best in different scenarios.

It might be useful to look back at the introduction and review each of these types. In this example, each of these types is signed.

In order to better understand these data types, write a program that subtracts two numbers using the three different data types described, and then compare the results.

To do this, print out `float1 - float2`, `double1 - double2`, and `int1 - int2`. Note that in the following code, "??" are placeholders for the numbers that will be used in the exercises below.

```
#include <iostream>

using namespace std;

float numFloat();
double numDouble();
int numInt();

int main() {
   cout << "float: " << numFloat() << endl;
   cout << "double: " << numDouble() << endl;
   cout << "int: " << numInt() << endl;
   return 0;
}

float numFloat() {
   float num1 = ??;
   float num2 = ??;
   return num1-num2;
}

double numDouble() {
   double num1 = ??;
   double num2 = ??;
   return num1-num2;
}

int numInt() {
   int num1 = ??;
   int num2 = ??;
   return num1-num2;
}
```

Note: Some compilers might report an error if a decimal is forced into an `int`. Remember type casting? (See endnote 2) If you try type casting the decimal into an `int`, your error will go away. Typecasting into an `int`: "`int num1 = (int)??;`"

1) You want to test out your calculator's limits. Try each of these functions subtracting the number 0.999999999998 from 1.0 (in the code above, set each `num1` to 1.0 and each `num2` to 0.999999999998).

a. What is the result when the numbers are represented as `float`s?

b. `double`s?

c. `int`s?

2) Now, going back to your physics calculation, try to see if your calculator can get a better answer than Google did for 1.0 - 0.999998.

a. What is the result when the numbers are represented as `float`s?

b. `double`s?

c. `int`s?

3) Modify your code to perform addition instead of subtraction.

4) Try adding these numbers: 100,000 and 0.5

a. What is the result when the numbers are represented as `float`s?

b. `double`s?

c. `int`s?

5) Try adding these numbers: 3.25 and 0.25

a. What is the result when the numbers are represented as `float`s?

b. `double`s?

c. `int`s?

6) Try these numbers: 222,333 and 1,222,333

a. What is the result when the numbers are represented as `float`s?

b. `double`s?

c. `int`s?

7) Now, try experimenting with more numbers. (I.e. large integers, extremely small decimals, decimals up to hundreds place, or decimals mixed with integers.)

a. Which of these numbers seemed to be best represented by `float`s?

b. `double`s?

c. `int`s?

d. Were your answers for `float`s and `double`s the same or different?

8) Compare your results to those of your neighbors to see if you got different answers to any of your calculations.

As you might have noticed, `int`s only work for integers, while `float`s and `double`s are better fitted for decimals. However, `floats` are represented differently from one computer to another. So you might or might not have seen your answers for `float`s differ to those of your neighbors.

9) In order to solve the mystery between `float`s and `double`s, it is important to look at how many bits of information each type stores. To do so, print out the size of `int`, `float`, and `double`. You can do this by using `sizeof()`, which prints the size of each of these data types in bytes.

```
int main() {
    cout << sizeof(int) << endl;
    cout << sizeof(float) << endl;
    cout << sizeof(double) << endl;
    return 0;
}
```

a. Given that 1 byte = 8 bits, how many bits does an `int` keep track of?

What about a `float`?

`double`?

If the size for `float`s and `double`s are the same on your computer, then the answers for your previous solutions (Previous exercises 1-7) were the same for `float`s and `double`s. On the flip side, if the size for `float`s and `double`s are different, your answers to your previous solutions were also different. This is because some machines store 32bits for the type `float`, while others will store 64bits.

b. Explain how storing 32 bits versus 64 bits for the type `float` will cause your answers to be different when doing arithmetic with small numbers.

3.3. Patriot Missile Failure

While some of these rounding errors seem to be very insignificant, as described by the accounting and Euro conversion examples, small errors can quickly add up. One of the most tragic events caused by rounding errors was the Patriot Missile Crisis in 1991.[5]


[6]

Patriot Missiles, which stand for Phased Array Tracking Intercept of Target, were originally designed to be mobile defenses against enemy aircraft. These missiles were supposed to explode right before encountering an incoming object. However, in Feb. 25, 1991, a Patriot Missile failed to intercept an incoming Iraqi Scud missile, which struck an army barrack in Dhahran. This killed 28 American soldiers and injured around 100 other people!

[5] Source: “The Patriot Missile Failure”, University of Minnesota. 4 Feb 2013. http://www.ima.umn.edu/~arnold/disasters/patriot.html.

[6] Source: [Photograph by officer or employee of United States Government]. 22 April 2006 [Photograph]. Patriot Missile PAC-2. Retrieved 5 March 2013. http://commons.wikimedia.org/wiki/File:Patriot_08.jpg#filelinks

What went wrong?

The system's internal clock recorded passage of time in tenths of seconds. However, as explained earlier, 1/10 has a non-terminating binary representation, which could lead to problems. Let's look into why this happens.

This is an approximation of what 1/10 looks like in binary:

0.0001100110011001100110011001100...

The internal clock used by the computer system only saved 24 bits. So this is what was being saved every 1/10 of a second:

0.00011001100110011001100

Chopping off any digits beyond the first 24 bits introduced an error of about:

0.0000000000000000000000011001100
which is about 0.000000095 seconds for each 1/10 second.

This Patriot battery had been running for about 100 hours before the incident. Imagine this error adding up 100 hours, 10 times for each second!

1) How does this small error add up in 100 hours? Calculate the total rounding error for this missile.

The small error for each 1/10 of a second was not believed to be a problem, for the Patriot Missiles were not supposed to be operated for more than 14 hours at a time. However, the Patriot battery had been running for over 100 hours, introducing an error of about 0.34s!

Given that the Patriot Missile was supposed to intercept a Scud traveling 1,624 meters per second, 0.34 seconds was a huge problem!

2) A Scud travels 1624 meters/second, how much would the Scud have advanced in the 0.34 seconds?

The Scud traveled more than half a kilometer in this amount of time. Therefore, the Scud was out of the Patriot Missiles range, hitting the army barrack in Dhahran.

3) Now that you know more about rounding errors, is there a way this catastrophe could have been prevented? What about trying to count steps by a different fraction? What would be an ideal fraction close to 1/10 that could have solved this problem?

3.3.1 Patriot Missile. Programming example.

Oh, how exciting! It looks like you have traveled back in time with your computer and you need to write the code for the Patriot Missile. You are using the same computer you have right now, and because you are doing this before taking this class, you decide to count time in 1/10 again.

```
int main() {
      float total = 0;

      //100 hours, 60 min/hr, 60 sec/min, 10 tenths/sec = 3600000
      for (int i = 0; i < 360000*10; i++){
            total += (float).1;
      }

      cout << total << " - total number of seconds recorded" <<  endl;

      int correct = 360000;
      cout << correct << " - the correct number of seconds in 100 hrs"
           << endl;

      float difference = correct - total;
      cout << difference << " - seconds your calculations are off"
           << endl;
      cout << 1624 * difference
           << " - meters traveled by scud in time difference" << endl;

      return 0;
}
```

1) Run this program. What is the distance traveled by the Scud in the amount of time the Patriot Missile was off?

2) You know better than this! You go back in time, again, and decide to re-write this code. What value are you going to use instead of 1/10?

3) What happens now? What is the amount traveled by the Scud in the amount of time your Patriot Missile calculated wrong?