*1.65 Prove that for each $n > 0$, a language B_n exists where

a. B_n is recognizable by an NFA that has n states, and

b. if $B_n = A_1 \cup \cdots \cup A_k$, for regular languages A_i, then at least one of the A_i requires a DFA with exponentially many states.

1.66 A ***homomorphism*** is a function $f\colon \Sigma \longrightarrow \Gamma^*$ from one alphabet to strings over another alphabet. We can extend f to operate on strings by defining $f(w) = f(w_1)f(w_2)\cdots f(w_n)$, where $w = w_1w_2\cdots w_n$ and each $w_i \in \Sigma$. We further extend f to operate on languages by defining $f(A) = \{f(w) \mid w \in A\}$, for any language A.

a. Show, by giving a formal construction, that the class of regular languages is closed under homomorphism. In other words, given a DFA M that recognizes B and a homomorphism f, construct a finite automaton M' that recognizes $f(B)$. Consider the machine M' that you constructed. Is it a DFA in every case?

b. Show, by giving an example, that the class of non-regular languages is not closed under homomorphism.

*1.67 Let the ***rotational closure*** of language A be $RC(A) = \{yx \mid xy \in A\}$.

a. Show that for any language A, we have $RC(A) = RC(RC(A))$.

b. Show that the class of regular languages is closed under rotational closure.

*1.68 In the traditional method for cutting a deck of playing cards, the deck is arbitrarily split two parts, which are exchanged before reassembling the deck. In a more complex cut, called Scarne's cut, the deck is broken into three parts and the middle part in placed first in the reassembly. We'll take Scarne's cut as the inspiration for an operation on languages. For a language A, let $CUT(A) = \{yxz \mid xyz \in A\}$.

a. Exhibit a language B for which $CUT(B) \neq CUT(CUT(B))$.

b. Show that the class of regular languages is closed under CUT.

1.69 Let $\Sigma = \{0,1\}$. Let $WW_k = \{ww \mid w \in \Sigma^* \text{ and } w \text{ is of length } k\}$.

a. Show that for each k, no DFA can recognize WW_k with fewer than 2^k states.

b. Describe a much smaller NFA for $\overline{WW_k}$, the complement of WW_k.

*1.70 We define the ***avoids*** operation for languages A and B to be

$$A \textit{ avoids } B = \{w \mid w \in A \text{ and } w \text{ doesn't contain any string in } B \text{ as a substring}\}.$$

Prove that the class of regular languages is closed under the *avoids* operation.

1.71 Let $\Sigma = \{0,1\}$.

a. Let $A = \{0^k u 0^k \mid k \geq 1 \text{ and } u \in \Sigma^*\}$. Show that A is regular.

b. Let $B = \{0^k 1 u 0^k \mid k \geq 1 \text{ and } u \in \Sigma^*\}$. Show that B is not regular.

1.72 Let M_1 and M_2 be DFAs that have k_1 and k_2 states, respectively, and then let $U = L(M_1) \cup L(M_2)$.

a. Show that if $U \neq \emptyset$, then U contains some string s, where $|s| < \max(k_1, k_2)$.

b. Show that if $U \neq \Sigma^*$, then U excludes some string s, where $|s| < k_1k_2$.

1.73 Let $\Sigma = \{0,1,\#\}$. Let $C = \{x\#x^{\mathcal{R}}\#x \mid x \in \{0,1\}^*\}$. Show that $\overline{C}$ is a CFL.