Sarvate–Beam designs: new existence results and large sets

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Abstract. A general construction for t–$SB(2t-1, 2t-2)$ designs is given. In addition, large sets of t–$SB(v, k)$ are discussed and some examples are provided.

1. Introduction

In general, an *SB design* is a block design in which every pair of points occurs in a different number of blocks. Below is a formal definition:

Definition 1. *A* Sarvate–Beam design $SB(v, k)$ *consists of a v–set V (called* points*) and a collection of k–subsets of V (called* blocks*) such that each distinct pair of elements in V occurs with different frequencies (i.e. in a different number of blocks). A* strict $SB(v, k)$ *is a SB design where for every i, $1 \le i \le \binom{v}{2}$, exactly one pair occurs i times.*

Example 1. *A strict $SB(4, 3)$ on $V = \{1, 2, 3, 4\}$ consists of the following blocks:* {1,2,4}, {1,3,4}, {1,3,4}, {2,3,4}, {2,3,4}, {2,3,4}, {2,3,4}. ▲

The general existence question of strict SB designs is still an open question. It is known that the necessary conditions for existence of strict SB designs are sufficient for $k = 3$ (see Dukes [**2**] and Ma, Chang and Feng [**5**]). Moreover, SB matrices have been studied by Dukes, Hurd and Sarvate [**3**]. The following definition appears in [**7**]:

Definition 2. *A t–$SB(v, k)$ design is a collection B of k–subsets of a v–set V such that each t–subset of V occurs a distinct number of times. In a strict t–SB design, for each i such that $1 \le i \le \binom{v}{t}$, there is exactly one t–subset which occurs in i blocks.*

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2. Existence results

Sarvate and Beam [**6**] showed that a non–strict $(n-2)$–$SB(n,n-1)$ design exists for every positive integer $n \geq 3$. Recently, Chan and Sarvate [**1**] have proven that a non–strict 2–$SB(n,n-1)$ design exists for every positive integer $n \geq 3$. In this paper, we want to generalize these two results and prove that a non–strict t–$SB(n,n-1)$ design exists for every positive integer n, where $n \geq 2t-1$ and $2 \leq t \leq n-2$. But first let us consider a general construction for non–strict t–$SB(2t-1,2t-2)$ designs for $t \geq 3$.

2.1. Construction of t–$SB(2t-1,2t-2)$ design for $t \geq 3$.

THEOREM 1. *A non–strict t–$SB(2t-1,2t-2)$ design exists for every positive integer $t \geq 3$.*

PROOF. Let B_i be the subset $\{1,2,\ldots,2t-1\}-\{i\}$ of the set $V = \{1,2,\ldots,2t-1\}$, and let the frequency of B_i, denoted $f(B_i)$, be 2^{i-1} for all i's. We claim that this construction produces a non–strict t–$SB(2t-1,2t-2)$ design. Suppose this is not true; that is, the above construction does not produce a non–strict t–$SB(2t-1,2t-2)$ design. It follows that there exists at least two subsets out of the $\binom{2t-1}{t}$ subsets that appear the same number of times. Let $a = \{a_1,a_2,\ldots,a_t\}$ and $b = \{b_1,b_2,\ldots,b_t\}$ be t–subsets such that $f(a) = f(b)$. Note that the frequency of each t–subset is determined by the the sum of $\binom{2t-1}{2t-2} - t = t-1$ blocks. We have

$$f(a) = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_{t-1}} \quad \text{and} \quad f(b) = 2^{j_1} + 2^{j_2} + \ldots + 2^{j_{t-1}}$$

Moreover, we have that

$$f(a) = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_{t-1}} = 2^{j_1} + 2^{j_2} + \ldots + 2^{j_{t-1}} = f(b)$$

Without loss of generality, assume that $i_k \neq j_n$ for $k,n = 1,2,\ldots,t-1$ (so that the equation above would be in its simplified form) and let i_1 be the smallest power. It follows that

$$1 + 2^{i_2-i_1} + \ldots + 2^{i_{t-1}-i_1} = 2^{j_1-i_1} + 2^{j_2-i_1} + \ldots + 2^{j_{t-1}-i_1}$$

Since there does not exist a j_k $(k = 1,2,\ldots,t-1)$ such that $j_k = i_1$, we have a contradiction. Therefore $f(a) = f(b)$ if and only if $a = b$.

■

NOTE 1. *The designs constructed here are not strict, as the size of the integer 2^{i-1} is sufficiently large. We see that there are t–subsets with frequencies greater than $2^{2t-2} = 4^{t-1}$ and it can be seen by mathematical induction that $4^{t-1} \geq \binom{2t-1}{t}$ for all positive integers t.*

Before we state the next theorem, recall that Chan and Sarvate [**1**] proved the following lemma:

LEMMA 1. *A non–strict t–$SB(n, n-1)$ design is also a non–strict $(t-1)$–$SB(n, n-1)$ design if $n-1 \geq 2t-2$.*

We now use this result to prove the following:

THEOREM 2. *A non–strict t–$SB(n, n-1)$ design exists for every positive integer n, where $n \geq 2t-1$ and $2 \leq t \leq n-2$.*

PROOF. It has been proven that for $t = 2$ and $t = n-2$, a non–strict t–$SB(n, n-1)$ design exists for positive integers $n \geq 3$. Now we only need to show that a non–strict t–$SB(n, n-1)$ design exists for $2 < t < n-2$. By Theorem 1, a non–strict t–$SB(2t-1, 2t-2)$ design exists for every positive integer $t \geq 3$. Furthermore, Lemma 1 together with Theorem 1 extend the existence of a non–strict t–$SB(n, n-1)$ to all n (for fixed t). Hence, the result follows. ■

3. Large sets of SB designs

In this section, we give examples of large sets of SB designs.

DEFINITION 3. *Let V be a v–set. A family of t–$SB(v, k)$ designs on V, say $B = \{B_1, B_2, \ldots, B_n\}$, is a* large set *with* multiplicity *s if the multiunion $\dot{\cup}_{i=1}^{n} B_i$ gives s copies of the set of all k–subsets of V for some integer s and if there is another family of t–$SB(v, k)$ designs $C = \{C_1, C_2, \ldots, C_m\}$ where $\dot{\cup}_{i=1}^{m} C_i$ gives u copies of the set of all k–subsets of V, then $s \leq u$.*

Before exhibiting examples of large sets of t–$SB(v, k)$, we need the following formula, which follows from counting in two ways the total of the block frequencies generated by the large set of t–$SB(v, k)$:

THEOREM 3. *Suppose the multiplicity for a large set for t–$SB(v, k)$ is s, and let the size of the large set be n. Then* $\boxed{s \cdot \binom{v}{k} = n \cdot \frac{\binom{v}{t}\left[\binom{v}{t}+1\right]}{2\binom{k}{t}}}$

EXAMPLE 2. *Consider the following set of 1–$SB(3, 2)$ (where the numbers in the table are block frequencies):*

Blocks	*Design 1*	*Design 2*
$\{1,2\}$	*0*	*2*
$\{1,3\}$	*1*	*1*
$\{2,3\}$	*2*	*0*

From Theorem 3, a large set of 1–$SB(3, 2)$ must satisfy $s = n$. Since there exists a unique 1–$SB(3, 2)$, and the largest block frequency is 2, it must be true that s is also at least 2. Hence, this example is a large set of 1–$SB(3, 2)$ with minimal $s = n = 2$. ▲

EXAMPLE 3. *Consider the following set of* 1–$SB(4,2)$*:*

Blocks	*Design 1*	*Design 2*	*Design 3*	*Design 4*	*Design 5*	*Design 6*
$\{1,2\}$	*0*	*1*	*2*	*2*	*0*	*0*
$\{1,3\}$	*0*	*0*	*0*	*1*	*2*	*2*
$\{1,4\}$	*1*	*0*	*0*	*0*	*2*	*2*
$\{2,3\}$	*2*	*2*	*0*	*0*	*0*	*1*
$\{2,4\}$	*0*	*0*	*2*	*2*	*1*	*0*
$\{3,4\}$	*2*	*2*	*1*	*0*	*0*	*0*

From Theorem 3, a large set of 1–$SB(4,2)$ *must satisfy* $6s = 5n$*. Since 6 divides n and 5 divides s, it must be true that the minimal* $s = 5$ *with corresponding* $n = 6$*. Hence, the example is a large set of* 1–$SB(4,2)$*.* ▲

EXAMPLE 4. *Consider the following set of* 1–$SB(7,2)$*:*

Blocks	*Design 1*	*Design 2*	*Design 3*
$\{1,2\}$	*0*	*1*	*1*
$\{1,3\}$	*0*	*1*	*1*
$\{1,4\}$	*0*	*1*	*1*
$\{1,5\}$	*0*	*2*	*0*
$\{1,6\}$	*0*	*0*	*2*
$\{1,7\}$	*1*	*0*	*1*
$\{2,3\}$	*0*	*1*	*1*
$\{2,4\}$	*0*	*0*	*2*
$\{2,5\}$	*0*	*1*	*1*
$\{2,6\}$	*1*	*0*	*1*
$\{2,7\}$	*1*	*0*	*1*
$\{3,4\}$	*0*	*2*	*0*
$\{3,5\}$	*1*	*1*	*0*
$\{3,6\}$	*1*	*1*	*0*
$\{3,7\}$	*1*	*1*	*0*
$\{4,5\}$	*1*	*1*	*0*
$\{4,6\}$	*1*	*0*	*1*
$\{4,7\}$	*2*	*0*	*0*
$\{5,6\}$	*2*	*0*	*0*
$\{5,7\}$	*1*	*1*	*0*
$\{6,7\}$	*1*	*0*	*1*

From Theorem 3, a large set of 1–$SB(7,2)$ *must satisfy* $3s = 2n$*. Since 3 divides n and 2 divides s, it must be true that the minimal* $s = 2$ *with* $n = 3$*. Hence, the example is a large set of* 1–$SB(7,2)$*.* ▲

EXAMPLE 5. *The family of* 2–$SB(6,3)$ *given in* [**1**] *contains errors in the last column, and should be replaced by the following:*

Blocks	*Design 1*	*Design 2*	*Design 3*	*Design 4*	*Design 5*
$\{1,2,3\}$	*0*	*0*	*2*	*4*	*4*
$\{1,2,4\}$	*0*	*1*	*1*	*5*	*3*
$\{1,2,5\}$	*1*	*0*	*3*	*1*	*5*
$\{1,2,6\}$	*0*	*2*	*3*	*3*	*2*
$\{1,3,4\}$	*0*	*4*	*0*	*4*	*2*
$\{1,3,5\}$	*1*	*1*	*2*	*1*	*5*
$\{1,3,6\}$	*1*	*5*	*1*	*2*	*1*
$\{1,4,5\}$	*2*	*3*	*0*	*2*	*3*
$\{1,4,6\}$	*1*	*5*	*0*	*3*	*1*
$\{1,5,6\}$	*2*	*3*	*3*	*0*	*2*
$\{2,3,4\}$	*1*	*1*	*1*	*5*	*2*
$\{2,3,5\}$	*2*	*0*	*4*	*0*	*4*
$\{2,3,6\}$	*2*	*1*	*5*	*1*	*1*
$\{2,4,5\}$	*3*	*0*	*2*	*2*	*3*
$\{2,4,6\}$	*3*	*2*	*2*	*3*	*0*
$\{2,5,6\}$	*3*	*1*	*5*	*0*	*1*
$\{3,4,5\}$	*5*	*2*	*1*	*1*	*1*
$\{3,4,6\}$	*4*	*4*	*0*	*2*	*0*
$\{3,5,6\}$	*4*	*2*	*4*	*0*	*0*
$\{4,5,6\}$	*5*	*3*	*1*	*1*	*0*

From Theorem 3, a large set of 2–$SB(6,3)$ *must satisfy* $s = 2n$. *Since there exist 16,444,250 nonisomorphic* 2–$SB(6,3)$ *(see* [**4**]*) with the smallest maximum block frequency therein being 5, it must be true that* s *is also at least 5. Since* s *is even, we see that the smallest possible* s *is* 6 *with corresponding* $n = 3$. *The above example was obtained by taking isomorphic copies of a single* 2–$SB(6,3)$ *design, and we claim that (using this particular design) the multiplicity cannot be less than 10. Hence, this example may or may not be a large set of* 2–$SB(6,3)$. ▲

Example 6. *Consider the following set of* 1–$SB(5,3)$*:*

Blocks	*Design 1*	*Design 2*	*Design 3*	*Design 4*
$\{1,2,3\}$	*0*	*0*	*0*	*2*
$\{1,2,4\}$	*0*	*0*	*0*	*2*
$\{1,2,5\}$	*0*	*1*	*1*	*0*
$\{1,3,4\}$	*0*	*0*	*2*	*0*
$\{1,3,5\}$	*1*	*0*	*0*	*1*
$\{1,4,5\}$	*0*	*0*	*2*	*0*
$\{2,3,4\}$	*0*	*2*	*0*	*0*
$\{2,3,5\}$	*0*	*2*	*0*	*0*
$\{2,4,5\}$	*2*	*0*	*0*	*0*
$\{3,4,5\}$	*2*	*0*	*0*	*0*

From Theorem 3, a large set of 1–$SB(5,3)$ *must satisfy* $2s = n$. *Since there exist 3 nonisomorphic* 1–$SB(5,3)$ *with the smallest max block frequency being 2, it must be true that s is also at least 2. Hence, the minimal possible s is* 2 *with corresponding* $n = 4$. *Thus, this example is a large set of* 1–$SB(5,3)$. ▲

4. Conclusion

The general problem of finding large sets of SB designs may be technical and difficult. However, producing an example of a large set of either 2–$SB(6,3)$ or of a 2–$SB(6,4)$ may be interesting. The reader is invited to produce such examples.

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