CSCE 310J
Data Structures & Algorithms

Dynamic programming
0-1 Knapsack problem

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CSCE 310J
Data Structures & Algorithms

- Giving credit where credit is due:
  - » Most of slides for this lecture are based on slides created by Dr. David Luebke, University of Virginia.
  - » Some slides are based on lecture notes created by Dr. Chuck Cusack, UNL.
  - » I have modified them and added new slides.

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Dynamic Programming

- Another strategy for designing algorithms is *dynamic programming*
  - » A metatechnique, not an algorithm (like divide & conquer)
  - » The word "programming" is historical and predates computer programming
- Use when problem breaks down into recurring small subproblems

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Dynamic programming

- It is used when the solution can be recursively described in terms of solutions to subproblems (*optimal substructure*).
- Algorithm finds solutions to subproblems and stores them in memory for later use.
- More efficient than "*brute-force methods*", which solve the same subproblems over and over again.

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Summarizing the Concept of Dynamic Programming

- Basic idea:
  - » Optimal substructure: optimal solution to problem consists of optimal solutions to subproblems
  - » Overlapping subproblems: few subproblems in total, many recurring instances of each
  - » Solve bottom-up, building a table of solved subproblems that are used to solve larger ones
- Variations:
  - » "Table" could be 3-dimensional, triangular, a tree, etc.

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Knapsack problem (Review)

Given some items, pack the knapsack to get the maximum total value. Each item has some weight and some value. Total weight that we can carry is no more than some fixed number W. So we must consider weights of items as well as their value.

Item #	Weight	Value
1	1	8
2	3	6
3	5	5

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Knapsack problem

There are two versions of the problem:

1. "0-1 knapsack problem" and
2. "Fractional knapsack problem"

1. Items are indivisible; you either take an item or not. Solved with *dynamic programming*
2. Items are divisible: you can take any fraction of an item. Solved with a *greedy algorithm*.
   - We have already seen this version

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0-1 Knapsack problem

- Given a knapsack with maximum capacity *W*, and a set *S* consisting of *n* items
- Each item *i* has some weight w_i and benefit value b_i (all w_i , b_i and *W* are integer values)
- Problem: How to pack the knapsack to achieve maximum total value of packed items?

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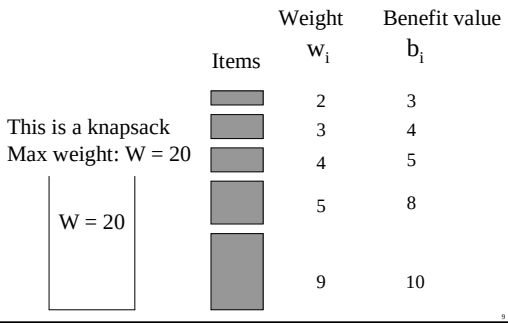
0-1 Knapsack problem: a picture



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0-1 Knapsack problem

- Problem, in other words, is to find
  $$\max \sum_{i \in T} b_i \text{ subject to } \sum_{i \in T} w_i \le W$$
- The problem is called a "*0-1*" problem, because each item must be entirely accepted or rejected.
- In the "*Fractional Knapsack Problem*," we can take fractions of items.

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0-1 Knapsack problem: brute-force approach

Let's first solve this problem with a straightforward algorithm

- Since there are *n* items, there are 2^n possible combinations of items.
- We go through all combinations and find the one with maximum value and with total weight less or equal to *W*
- Running time will be $O(2^n)$

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0-1 Knapsack problem: brute-force approach

- Can we do better?
- Yes, with an algorithm based on dynamic programming
- We need to carefully identify the subproblems

Let's try this:
If items are labeled *1..n*, then a subproblem would be to find an optimal solution for
S_k = *{items labeled 1, 2, .. k}*

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Defining a Subproblem

If items are labeled *1..n*, then a subproblem would be to find an optimal solution for S_k = *{items labeled 1, 2, .. k}*

- This is a reasonable subproblem definition.
- The question is: can we describe the final solution (S_n) in terms of subproblems (S_k)?
- Unfortunately, we can't do that.

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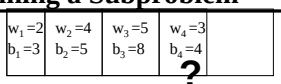
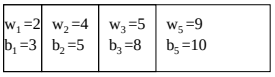
Defining a Subproblem



?

Max weight: W = 20
For S_4:
Total weight: 14;
Maximum benefit: 20



For S_5:
Total weight: 20
Maximum benefit: 26

Item #	Weight w_i	Benefit b_i
1	2	3
2	3	4
3	4	5
4	5	8
5	9	10

S_4

S_5

Solution for S_4 is not part of the solution for S_5!!!

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Defining a Subproblem (continued)

- As we have seen, the solution for S_4 is not part of the solution for S_5
- So our definition of a subproblem is flawed and we need another one!
- Let's add another parameter: *w*, which will represent the exact weight for each subset of items
- The subproblem then will be to compute *B[k,w]*

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Recursive Formula for subproblems

Recursive formula for subproblems:

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{B[k-1,w], B[k-1,w-w_k]+b_k\} & \text{else} \end{cases}$$

It means, that the best subset of S_k that has total weight *w* is:

1) the best subset of S_{k-1} that has total weight *w*, **or**
2) the best subset of S_{k-1} that has total weight w-w_k plus the item *k*

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Recursive Formula

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{B[k-1,w], B[k-1,w-w_k]+b_k\} & \text{else} \end{cases}$$

- The best subset of S_k that has the total weight *w*, either contains item *k* or not.
- First case: $w_k > w$. Item *k* can't be part of the solution, since if it was, the total weight would be $> w$, which is unacceptable.
- Second case: $w_k \leq w$. Then the item *k* can be in the solution, and we choose the case with greater value.

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0-1 Knapsack Algorithm

```
for w = 0 to W
    B[0,w] = 0
for i = 1 to n
    B[i,0] = 0
for i = 1 to n
    for w = 0 to W
        if wi <= w // item i can be part of the solution
            if bi + B[i-1,w-wi] > B[i-1,w]
                B[i,w] = bi + B[i-1,w- wi]
            else
                B[i,w] = B[i-1,w]
        else B[i,w] = B[i-1,w]  // wi > w
```

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Running time

```
for w = 0 to W                    O(W)
    B[0,w] = 0
for i = 1 to n
    B[i,0] = 0
for i = 1 to n                    Repeat n times
    for w = 0 to W                O(W)
        < the rest of the code >
```

What is the running time of this algorithm?

O(n*W)

Remember that the brute-force algorithm takes $O(2^n)$

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Example

Let's run our algorithm on the following data:

n = 4 (# of elements)
W = 5 (max weight)
Elements (weight, benefit):
(2,3), (3,4), (4,5), (5,6)

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Example (2)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1						
2						
3						
4						

```
for w = 0 to W
    B[0,w] = 0
```

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Example (3)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0					
2	0					
3	0					
4	0					

```
for i = 1 to n
    B[i,0] = 0
```

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Example (4)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0				
2	0					
3	0					
4	0					

i=1
b_i=3
w_i=2
w=1
w-w_i =-1

```
if w_i <= w // item i can be part of the solution
    if b_i + B[i-1,w-w_i] > B[i-1,w]
        B[i,w] = b_i + B[i-1,w- w_i]
    else
        B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w]  // w_i > w
```

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Example (5)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	**3**			
2	0					
3	0					
4	0					

i=1
b_i=3
w_i=2
w=2
w-w_i =0

```
if w_i <= w // item i can be part of the solution
    if b_i + B[i-1,w-w_i] > B[i-1,w]
        B[i,w] = b_i + B[i-1,w- w_i]
    else
        B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w]  // w_i > w
```

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Example (6)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	**3**		
2	0					
3	0					
4	0					

i=1
b_i=3
w_i=2
w=3
w-w_i =1

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

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Example (7)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	**3**	
2	0					
3	0					
4	0					

i=1
b_i=3
w_i=2
w=4
w-w_i =2

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

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Example (8)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	**3**
2	0					
3	0					
4	0					

i=1
b_i=3
w_i=2
w=5
w-w_i =3

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

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Example (9)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	**0**				
3	0					
4	0					

i=2
b_i=4
w_i=3
w=1
w-w_i =-2

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else **B[i,w] = B[i-1,w]** // w_i > w

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Example (10)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	**3**			
3	0					
4	0					

i=2
b_i=4
w_i=3
w=2
w-w_i =-1

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else **B[i,w] = B[i-1,w]** // w_i > w

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Example (11)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	**4**		
3	0					
4	0					

i=2
b_i=4
w_i=3
w=3
w-w_i =0

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

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Example (12)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	**4**	
3	0					
4	0					

$i=2$
$b_i=4$
$w_i=3$
$w=4$
$w-w_i=1$

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

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Example (13)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	**7**
3	0					
4	0					

$i=2$
$b_i=4$
$w_i=3$
$w=5$
$w-w_i=2$

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

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Example (14)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	**0**	**3**	**4**		
4	0					

$i=3$
$b_i=5$
$w_i=4$
$w= 1..3$

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else **B[i,w] = B[i-1,w]** // w_i > w

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Example (15)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	**5**	
4	0					

$i=3$
$b_i=5$
$w_i=4$
$w= 4$
$w- w_i=0$

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

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Example (16)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	**7**
4	0					

$i=3$
$b_i=5$
$w_i=4$
$w= 5$
$w- w_i=1$

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // w_i > w

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Example (17)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	**0**	**3**	**4**	**5**	

$i=4$
$b_i=6$
$w_i=5$
$w= 1..4$

if w_i <= w // item i can be part of the solution
  if b_i + B[i-1,w-w_i] > B[i-1,w]
    B[i,w] = b_i + B[i-1,w- w_i]
  else
    B[i,w] = B[i-1,w]
else **B[i,w] = B[i-1,w]** // w_i > w

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Example (18)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

i=4
b_i=6
w_i=5
w= 5
w- w_i=0

```
if w_i <= w // item i can be part of the solution
      if b_i + B[i-1,w-w_i] > B[i-1,w]
         B[i,w] = b_i + B[i-1,w- w_i]
      else
         B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w]  // w_i > w
```

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Comments

- This algorithm only finds the max possible value that can be carried in the knapsack
  - » I.e., the value in B[n,W]
- To know the items that make this maximum value, an addition to this algorithm is necessary.

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How to find actual Knapsack Items

- All of the information we need is in the table.
- *B*[*n,W*] is the maximal value of items that can be placed in the Knapsack.
- Let i=n and k=W

if $B[i,k] \neq B[i-1,k]$ then
  mark the i^{th} item as in the knapsack
  $i = i-1$, $k = k-w_i$
else
  $i = i-1$ // Assume the i^{th} item is <u>not</u> in the knapsack
  // Could it be in the optimally packed knapsack?

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Finding the Items

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

i=4
k= 5
b_i=6
w_i=5
$B[i,k]$ = 7
$B[i-1,k]$ =7

```
i=n, k=W
while i,k > 0
    if B[i,k] ≠ B[i–1,k] then
        mark the i^th item as in the knapsack
        i = i–1, k = k-w_i
    else
        i = i–1
```

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Finding the Items (2)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

i=4
k= 5
b_i=6
w_i=5
$B[i,k]$ = 7
$B[i-1,k]$ =7

```
i=n, k=W
while i,k > 0
    if B[i,k] ≠ B[i–1,k] then
        mark the i^th item as in the knapsack
        i = i–1, k = k-w_i
    else
        i = i–1
```

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Finding the Items (3)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

i=3
k= 5
b_i=6
w_i=4
$B[i,k]$ = 7
$B[i-1,k]$ =7

```
i=n, k=W
while i,k > 0
    if B[i,k] ≠ B[i–1,k] then
        mark the i^th item as in the knapsack
        i = i–1, k = k-w_i
    else
        i = i–1
```

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Finding the Items (4)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

i=2
k= 5
b_i=4
w_i=3
$B[i,k]$ = 7
$B[i–1,k]$ =3
k – w_i=2

i=n, k=W
while i,k > 0
  if $B[i,k] \neq B[i–1,k]$ then
    mark the i^{th} item as in the knapsack
    $i = i–1, k = k-w_i$
  else
    i = i–1

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Finding the Items (5)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

i=1
k= 2
b_i=3
w_i=2
$B[i,k]$ = 3
$B[i–1,k]$ =0
k – w_i=0

i=n, k=W
while i,k > 0
  if $B[i,k] \neq B[i–1,k]$ then
    mark the i^{th} item as in the knapsack
    $i = i–1, k = k-w_i$
  else
    i = i–1

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Finding the Items (6)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

i=0
k= 0

The optimal knapsack should contain {1, 2}

i=n, k=W
while i,k > 0
  if $B[i,k] \neq B[i–1,k]$ then
    mark the n^{th} item as in the knapsack
    $i = i–1, k = k-w_i$
  else
    $i = i–1$

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Finding the Items (7)

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i\W	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

The optimal knapsack should contain {1, 2}

i=n, k=W
while i,k > 0
  if $B[i,k] \neq B[i–1,k]$ then
    mark the n^{th} item as in the knapsack
    $i = i–1, k = k-w_i$
  else
    $i = i–1$

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Review: The Knapsack Problem And Optimal Substructure

- Both variations exhibit optimal substructure
- To show this for the 0-1 problem, consider the most valuable load weighing at most *W* pounds
  - » *If we remove item j from the load, what do we know about the remaining load?*
  - » A: remainder must be the most valuable load weighing at most $W - w_j$ that thief could take, excluding item *j*

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Solving The Knapsack Problem

- The optimal solution to the fractional knapsack problem can be found with a greedy algorithm
  - » *Do you recall how?*
  - » Greedy strategy: take in order of dollars/pound
- The optimal solution to the 0-1 problem cannot be found with the same greedy strategy
  - » Example: 3 items weighing 10, 20, and 30 pounds, knapsack can hold 50 pounds
    - ❖ *Suppose item 2 is worth $100. Assign values to the other items so that the greedy strategy will fail*

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The Knapsack Problem: Greedy Vs. Dynamic

- The fractional problem can be solved greedily
- The 0-1 problem can be solved with a dynamic programming approach

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Memoization

- *Memoization* is another way to deal with overlapping subproblems in dynamic programming
  - » After computing the solution to a subproblem, store it in a table
  - » Subsequent calls just do a table lookup
- With memoization, we implement the algorithm recursively:
  - » If we encounter a subproblem we have seen, we look up the answer
  - » If not, compute the solution and add it to the list of subproblems we have seen.
- Must useful when the algorithm is easiest to implement recursively
  - » Especially if we do not need solutions to all subproblems.

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Conclusion

- Dynamic programming is a useful technique of solving certain kind of problems
- When the solution can be recursively described in terms of partial solutions, we can store these partial solutions and re-use them as necessary (memoization)
- Running time of dynamic programming algorithm vs. naïve algorithm:
  - » 0-1 Knapsack problem: **O(W*n)** vs. **$O(2^n)$**

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