Divide & Conquer

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1. Divide instance of problem into two or more smaller instances
2. Solve smaller instances recursively
3. Obtain solution to original (larger) instance by combining these solutions

Divide and conquer technique

General Divide and Conquer recurrence:

\[ T(n) = aT(n/b) + f(n) \quad \text{where } f(n) \in \Theta(n^k) \]

1. \( a < b^k \quad T(n) \in \Theta(n^k) \)
2. \( a = b^k \quad T(n) \in \Theta(n^k \log n) \)
3. \( a > b^k \quad T(n) \in \Theta(n^{\log_b a}) \)

Note: the same results hold with \( O \) instead of \( \Theta \).
Mergesort

Algorithm:
1. Split array $A[1..n]$ in two and make copies of each half in arrays $B[1..n/2]$ and $C[1..n/2]$.
2. Sort arrays $B$ and $C$.
3. Merge sorted arrays $B$ and $C$ into array $A$ as follows:
   - Repeat the following until no elements remain in one of the arrays:
     - Compare the first elements in the remaining unprocessed portions of the arrays.
     - Copy the smaller of the two into $A$, while incrementing the index indicating the unprocessed portion of that array.
   - Once all elements in one of the arrays are processed, copy the remaining unprocessed elements from the other array into $A$.

Mergesort Examples

Animated Example:
http://www.cs.hope.edu/~alganim/ animator/Animator.html
Another animated example:
http://math.hws.edu/ TMCM/ java/xSortLab/index.html

Merging Sorted Sequences

Problem:
- Given two sequences $A$ and $B$ sorted in nondecreasing order, merge them to create one sorted sequence $C$.

Strategy:
- Determine the first item in $C$.
- It is the minimum between the first items of $A$ and $B$.
  - Suppose it is the first item of $A$.
  - Then, rest of $C$ consisting of merging rest of $A$ with $B$.

Algorithm: Merge

4. Merge($A$, $B$, $C$)
   - if (A is empty)
     - rest of $C$ = rest of $B$
   - else if (B is empty)
     - rest of $C$ = rest of $A$
   - else if (first of $A$ < first of $B$)
     - first of $C$ = first of $A$
     - merge (rest of $A$, $B$, rest of $C$)
   - else
     - first of $C$ = first of $B$
     - merge ($A$, rest of $B$, rest of $C$)
   - return

W(n) = n – 1
**Run-time**: The number of basic operations performed (e.g., compare and swap)

**Memory**: The amount of memory used beyond what is needed to store the data being sorted
- “In place” algorithms use a constant amount of extra memory—the constant may be zero
- Other algorithms are described as linear or exponential with respect to the space used.
- Less is better, but there is often a space/time trade-off.

**Stability**: An algorithm is stable if it preserves the relative order of equal keys

- Merge sort always partitions the array equally.
- Thus, the recursive depth is always $O(\log n)$
- The amount of work done at each level is $O(n)$
- Thus, the complexity should be $O(n \log n)$
- Once again we have,
  - $T(n) \leq 2T(n/2) + O(n)$
- The amount of extra memory used is $O(n)$
- Note: Merge sort is stable

**Efficiency of merge sort**
- All cases have same efficiency: $\Theta(n \log n)$
- Number of comparisons is close to theoretical minimum for comparison-based sorting:
  - $\lceil \log n \rceil \approx \lceil n \log n - 1.44 \rceil$
- Space requirement: $\Theta(n)$ (NOT in-place)
- Can be implemented without recursion (bottom-up)

**Quick sort by Hoare (1962)**
- Select a pivot (partitioning element)
- Rearrange the list so that all the elements in the positions before the pivot are smaller than or equal to the pivot and those after the pivot are larger than the pivot (See algorithm Partition in section 4.2)
- Exchange the pivot with the last element in the first (i.e., ≤ sublist) – the pivot is now in its final position
- Sort the two sublists recursively

**Quick sort**

- Animated Example: [http://www.cs.hope.edu/~alganim/animator/Animator.html](http://www.cs.hope.edu/~alganim/animator/Animator.html)
- Another animated example: [http://math.hws.edu/TMCM/java/sSortLab/index.html](http://math.hws.edu/TMCM/java/sSortLab/index.html)
Design and Analysis of Algorithms  

Chapter 4

QuickSort Example

Recursive implementation with the left most array entry selected as the pivot element.

```
<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5</td>
<td>2</td>
<td>1</td>
<td>7</td>
<td>6</td>
<td>8</td>
<td>9</td>
<td>4</td>
<td>0</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>
```

QuickSort Algorithm

- **Input:** Array E and indices first, and last, s.t. elements E[] are defined for first ≤ i ≤ last
- **Output:** E[first], ..., E[last] is a sorted rearrangement of the array
- **Void quickSort(Element[] E, int first, int last)**
  - if (first < last)
    - Element pivotElement = E[first];
    - Key pivot = pivotElement.Key;
    - int splitPoint = partition(E, pivot, first, last);
    - E[splitPoint] = pivotElement;
    - quickSort(E, first, splitPoint - 1);
    - quickSort(E, splitPoint + 1, last);
  - return;

QuickSort Analysis

- Partition can be done in O(n) time, where n is the size of the array
- Let T(n) be the number of compares required by QuickSort
- If the pivot ends up at position k, then we have
  - T(n) = T(n−k) + T(k−1) + n
- To determine best-, worst-, and average-case complexity we need to determine the values of k that correspond to these cases.

Best-Case Complexity

- The best case is clearly when the pivot always partitions the array equally.
- Intuitively, this would lead to a recursive depth of at most \( \log n \) calls
- We can actually prove this. In this case
  - \( T(n) \leq T(n/2) + T(n/2) + n \leq \Theta(n \log n) \)

Worst-Case and Average-Case Complexity

- The worst-case is when the pivot always ends up in the first or last element. That is, partitions the array as unequally as possible.
- In this case
  - \( T(n) = T(n−1) + T(1−1) + n = T(n−1) + n = n + (n−1) + \ldots + 1 = n(n + 1)/2 \leq \Theta(n^2) \)
- Average case is rather complex, but is where the algorithm earns its name. The bottom line is:
  - \( \Theta(n \log n) \)

Summary of quicksort

- **Best case:** split in the middle → \( \Theta(n \log n) \)
- **Worst case:** sorted array! → \( \Theta(n^2) \)
- **Average case:** random arrays → \( \Theta(n \log n) \)

- Improvements:
  - better pivot selection: median of three partitioning avoids worst case in sorted files
  - switch to insertion sort on small subfiles
  - elimination of recursion these combine to 20-25% improvement
- Considered the method of choice for internal sorting for large files (\( n \geq 10000 \))
Inspired by Quicksort compute Convex Hull:

1. Assume points are sorted by x-coordinate values
2. Identify extreme points $P_1$ and $P_2$ (part of hull)
3. Compute upper hull:
   - Find point $P_{\text{max}}$ that is farthest away from line $P_1P_2$
   - Compute the hull of the points to the left of line $P_1P_{\text{max}}$
   - Compute the hull of the points to the left of line $P_{\text{max}}P_2$
4. Compute lower hull in a similar manner

Finding point farthest away from line $P_1P_2$ can be done in linear time.

This gives same efficiency as quicksort:
- \textit{Worst case}: $\Theta(n^2)$
- \textit{Average case}: $\Theta(n \log n)$

If points are not initially sorted by x-coordinate value, this can be accomplished in $\Theta(n \log n)$ — no increase in asymptotic efficiency class.

Other algorithms for convex hull:
- Graham's scan
- DCHull
- also in $\Theta(n \log n)$