CSCE 310J
Data Structures & Algorithms

Recursion and Induction

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- Giving credit where credit is due:
  - » Most of slides for this lecture are based on slides created by Dr. Ben Choi, Louisiana Technical University.
  - » I have modified them and added new slides

Recursive Procedures

- You were supposed to be introduced to recursive procedures in CSCE 156.
- What recursive procedures have you seen?
- Many loops can be replaced with recursive procedures.
  - » In some algorithms, it is easier to use recursion than loops!
- Divide and Conquer algorithms frequently use recursive procedures to divide the data set into one or more parts and then recursively apply the algorithm to the smaller parts.

Example:
Binary Search

```
int binarySearch(int[] entry, int first, int last, int key)
if (last < first)
    index = -1
else
    int middle = (first + last)/2
    if (key == entry[middle])
      index = middle
    else if (key < entry[middle])
      index = binarySearch(entry, first, middle -1, key)
    else
      index = binarySearch(entry, middle +1, last, key)
return index
```


Designing Recursive Procedures

- Think Inductively
- Converge to a base case (stopping the recursion)
  - » identify some unit of measure (running variable)
  - » identify the *easy* cases, called base cases
- Assume algorithm **p** must solve the problem with input sizes ranging from 0 through 100
  - » assume **p99** solved a subproblem for all sizes 0 through 99
  - » if **p** detects a case that is not the base case, it calls **p99** with a proper subset of the input data
- **p99** satisfies:
  1. The subproblem size is less than **p**'s problem size
  2. The subproblem size is not below the minimum
  3. The subproblem satisfies all other preconditions of **p99** (which are the same as the preconditions of **p**)

Recursive Procedure Design Example

- Problem:
  - » Write a delete(L, x) procedure for a list L, which is supposed to delete the first occurrence of x
  - » However, it is possible x does not occur in L
- Strategy:
  - » Use a recursive Procedure
  - » The size of the problem is the number of elements in list L
  - » Use IntList ADT
  - » Base cases: ??
  - » Running variable (converging number): ??

ADT for Lists

IntList nil //constant denoting the empty list.

IntList consructList(int newElement, IntList oldList)
Precondition: None.
Postconditions: If newList = constructList(newElement, oldList) then
1. newList refers to a newly created list object;
2. newList ≠ nil;
3. first(newList) = newElement;
4. rest(newList) = oldList

int first(IntList aList) // access function
Precondition: aList ≠ nil
Postcondition: if element = first(aList) then
1. element ≠ nil

IntList rest(IntList aList) // access fcn
Precondition: aList ≠ nil

Algorithm for Recursive delete(L, x) from list

```
IntList delete(IntList initialList, int anElement)
  IntList resultList, subproblemList;
    if (initialList == nil)
      resultList = initialList;
    else if (anElement == first(initialList))
      resultList = rest(initialList);
    else
      subproblemList = delete99(rest(initialList), anElement);
      resultList = constructList(first(initialList), subproblemList);
  return resultList;
```

Now remove “99” from the called subroutine. That is, change delete99() to delete().

Algorithm for non-recursive delete(L, x)

```
IntList delete(IntList L, int x)
  IntList newL, tempL;
  tempL = L; newL = nil;
  // search for x, copying elements to newL until x is found or tempL is empty
  while (tempL != nil && x != first(tempL))
       newL = constructList(first(tempL), newL); //copy element
       tempL = rest(tempL);                      // skip copied element
  If (tempL != nil)                              // ⇒ x == first(tempL)
    tempL = rest(tempL);                         // remove x
  while (tempL != nil)                           // copy remaining elements
       newL = cons(first(tempL), newL);
       tempL = rest(tempL);
  return newL;                                   // x is not in newL
```


Convert a non-recursive procedure to a recursive procedure

- Change the procedure with a loop to call a recursive procedure without a loop
- Recursive Procedure begins by acting like a WHILE loop
  - While(Not Base Case)
  - Set up Sub-problem
  - Recursive call to continue
- The recursive function may need an additional parameter
  - E.g., to replace an *index* in a FOR loop of the non-recursive procedure.

Transforming loop into a recursive procedure

- Local variables within the loop body
  - give the variable only one value in any one pass
  - for variables that must be updated, do all the updates at the end of the loop body
- Re-expressing a while loop with recursion
  - Additional parameters
    - Variables updated in the loop become procedure input parameters. Their *initial values* at loop entry correspond to the actual parameters in the top-level call of the recursive procedure.
    - Variables referenced in the loop but not updated may also become parameters
  - The recursive procedure begins by mimicking the while condition and returns if the while condition is false
    - a break also corresponds to a procedure return
  - Continue by updating variables and make the recursive call

Removing While Loop Example

```
int factLoop(int n)
  int k=1; int f = 1

  while (k ≤ n)
    int fnew = f*k;
    int knew = k+1;
    k = knew; f = fnew;
return f;
```

```
int factLoop(int n)
  return factRec(n, 1, 1);

int factRec(int n, int k, int f)
if (k ≤ n)
    int fnew = f*k;
    int knew = k+1
    f = factRec(n, knew, fnew)
return f;
```

Removing For Loop Example

Convert the following sequentialSearch() procedure to a recursive procedure without a loop

```
int sequentialSearch(int[] entry, int nEntries, int key)
1. int answer, index;
2. answer = -1; // Assume failure.
3. for (index = 0; index < nEntries; index++)
4.     if (key == entry[index])
5.         answer = index; // Success!
6.         break; // Done!
7. return answer;
```

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Recursive Procedure without loops e.g.

Call with: sequentialSearchRecursive(entry, 0, nEntries, key)

```
seqSearchRecursive(int[] entry, int index, int nEntries, int key)
0. int answer;
1. if (index ≥ nEntries)
2.   answer = -1;
3. else if (entry[index] == key)  // index < nEntries
4.   answer = index;
5. else
6.   answer = sequentialSearchRecursive(entry, index+1, nEntries, key);
7. return answer;
```

- Compare to: for (index = 0; index < nEntries; index++)

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Analyzing Recursive Procedure using Recurrence Equations

- Let n be the size of the problem
- Worst-Case Analysis (for procedure with no loops)
  - » T(n) = the individual cost for a sequence of blocks
    + the maximum cost for an alternation of blocks
    + the cost of subroutine call, S(f(n))
    + the cost of recursive procedure call, T(g(n))
- e.g. sequentialSearchRecursive(),
  - » Basic operation is comparison of array element, cost 1
  - » 1. + max(2., (3. + max(4., (5. + 6.))) + (7.)
  - » 0 + max(0, (1 + max(0, (0+T(n-1))) + 0
  - » $T(n) = T(n-1) + 1$; $T(0) = 0$
  - » => $T(n) = n$; $T(n) \in \theta(n)$

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Consider binarySearch()

```
int binarySearch(int[] entry, int first, int last, int key)
1.  if (last < first)
2.      index = -1
3.  else
4.      int middle = (first + last)/2
5.      if (key == entry[middle])
6.        index = middle
7.      else if (key < entry[middle])
8.        index = binarySearch(entry, first, middle -1, key)
9.      else
10.       index = binarySearch(entry, middle +1, last, key)
11. return index
```

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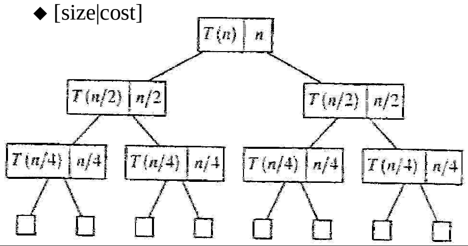
Evaluate recursive equation using Recursion Tree

Does this equation apply to binarySearch()?

- Evaluate: $T(n) = T(n/2) + T(n/2) + n$
  - » Working copy: $T(k) = T(k/2) + T(k/2) + k$
  - » For $k=n/2$, $T(n/2) = T(n/4) + T(n/4) + (n/2)$
- [size|cost]



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Recursion Tree e.g.

- To evaluate the total cost of the recursion tree
  - » sum all the non-recursive costs of all nodes
  - » = Sum (rowSum(cost of all nodes at the same depth))
- Determine the maximum depth of the recursion tree:
  - » For our example, at tree depth d, the size parameter is $n/(2^d)$
  - » the size parameter converges to the base case, i.e. case 1 where $n/(2^d) = 1 \Rightarrow d = \lg(n)$
- The rowSum for each row is n
- Therefore, the total cost, $T(n) = n \lg(n)$

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Proving Correctness of Procedures: *Proof*

- What is a Proof?
  - » A Proof is a *sequence* of statements that form a logical argument.
  - » Each statement is a complete sentence in the normal grammatical sense.
- Each statement should draw a new conclusion *from*:
  - » *axiom*: well known facts
  - » *assumptions*: premises of the theorem you are proving or inductive hypothesis
  - » *intermediate conclusions*: statements established earlier
- *To* arrive at the last statement of a proof that must be the conclusion of the proposition being proven

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Format of Theorem, Proof Format

- A proposition (theorem, lemma, and corollary) is represented as:

  $\forall x \in W (A(x) => C(x))$

  for all x in W, if A(x) then C(x)
  - » the set W is called world,
  - » A(x) represents the *assumptions*
  - » C(x) represents the *conclusion*, the goal statement
  - » => is read as "implies"
- Proof sketches provide an outline of a proof
  - » the strategy, the road map, or the plan.
- Two-Column Proof Format
  - » Statement : Justification (supporting facts)

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Induction Proofs

- Induction proofs are a mechanism, often the only mechanism, for proving a statement about an infinite set of objects.
  - » Inferring a property of a set based on the property of its objects
- Induction is often done *over* the set of natural numbers {0,1,2,…}
  - » starting from 0, then 1, then 2, and so on
- Induction is valid over a set, provided that:
  - » The set is partially ordered;
    - ❖ i.e. an order relationship is defined between some pairs of elements, but perhaps not between all pairs.
  - » There is no infinite chain of decreasing elements in the set. (e.g. cannot be set of all integers)

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Induction Proof Schema

Prove: $\forall x \in W (A(x) => C(x))$

- Proof:
  1. The Proof is by induction on x, <description of x>
  2. The base case is, cases are, <base-case>
  3. <Proof of goal statement with base-case substituted into it, that is, C(base-case)>
  4. For <x> greater than <base-case>, assume that A(y) => C(y) holds for all $y \in W$ such that y < x.
  5. <Proof of the goal statement, C(x), exactly as it appears in the proposition>.

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Induction Proof Example

- Prove:

  For all $n \geq 0$, $\sum_{i=0}^{n} i(i+1)/2 = n(n+1)(n+2)/6$
- Proof: …
  - » Left as an exercise for the student ☺

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Proving Correctness of Procedures

- *Things should be made as simple as possible – but not simpler*
  - » Albert Einstein
- Proving Correctness of procedures is a difficult task in general; the trick is to *make it as simple as possible*.
  - » No loops are allowed in the procedure!
  - » Variable is assigned a value only once!
- Loops are converted into Recursive procedures.
- Additional variables are used to make single-assignment (write-once read many) possible.
  - » x = y+1 does imply the equation x = y+1 for entire time

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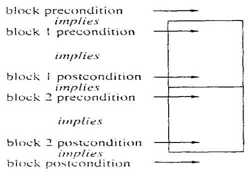
General Correctness Lemma

- If all *preconditions* hold when the block is entered,
- then all *postconditions* hold when the block exits
- And, the procedure will terminate!
  - » Chains of Inference: Sequence



Proving Correctness of binarySearch()

```
int binarySearch(int[] entry, int first, int last, int key)
1.  if (last < first)
2.      index = -1
3.  else
4.     int middle = (first + last)/2
5.     if (key == entry[middle])
6.        index = middle
7.     else if (key < entry[middle])
8.        index = binarySearch(entry, first, middle -1, key)
9.     else
10.       index = binarySearch(entry, middle +1, last, key)
11. return index
```


Proving Correctness of Binary Search

- Lemma   *(preconditions => postconditions)*
  - » *if* binarySearch(*entry*, *first*, *last*, *key*) is called, and
    the problem size is $n = (last - first + 1)$, for all $n \geq 0$, and
    entry[*first*], … *entry*[*last*] are in nondecreasing order,
  - » *then* it returns –1 if *key* does not occur in *entry* within the range *first*, …, *last*, and
    it returns *index* such that *key*=*entry*[*index*] otherwise
- Proof
  - » The proof is by induction on *n*, the problem size.
  - » The base case in $n = 0$.
  - » In this case, line 1 is true, line 2 is reached, and –1 is returned. *(the postcondition is true)*

Inductive Proof, continue

- For $n > 0$, assume that binarySearch(*entry*, *f*, *l*, *key*) satisfies the lemma on problems of size *k*, such that
  $0 \leq k < n$, and *f* and *l* are any indices such that $k = l - f + 1$
  - » For $n > 0$, line 1 is false, … *middle* is within the search range ($first \leq middle \leq last$).
  - » If line 5 is true, the procedure *terminates* with $index = middle$. (the postcondition is true)
  - » If line 5 is false, from ($first \leq middle \leq last$) and def. of *n*,
    $(middle - 1) - first + 1 \leq (n - 1)$
    $last - (middle + 1) + 1 \leq (n - 1)$
  - » so the inductive hypothesis applies for both recursive calls,
  - » If line 7 is true, … the preconditions of binarySearch are satisfied, we can assume that the call accomplishes the objective.
  - » If line 8 returns a positive *index*, done.
  - » If line 8 returns –1, this implies that *key* is not in *entry* in the *first* … *middle*-1, also since line 7 is true, *key* is not in *entry* in range *min*… *last*, so returning – 1 is correct (done).
  - » If line 7 is false, … similarly the postconditions are true. (done!)