

Predicates and Quantifiers

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Predicate in English

- In English, a sentence has 2 parts: the subject and the predicate.
- The predicate is the part of the sentence that states something about the subject.
- For example, in the sentence
 “John Cusack was in the movie *Say Anything*,”
John Cusack is the subject, and was in the movie *Say Anything* is the predicate.

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Predicate in Logic

- In logic, a predicate is something that is affirmed or denied about the argument of a proposition.
- For example, in the proposition
 “ x was in the movie *Say Anything*,”
 x is the argument, and was in the movie *Say Anything* is the predicate.
- In logic, however, the argument does not always correspond to the subject. For instance, in the proposition
 “John Cusack was in the movie x ,”
 x is the argument, and John Cusack was in the movie is the predicate. You might prefer to word the predicate as is a movie that John Cusack was in.

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Propositional Functions: Prelude

- Recall that a proposition is a statement that is either true or false.
 - You can think of a function as a “black box” which takes one or more inputs, and produces a single output.
 - You may have learned that a function has a domain (the set of possible inputs) and a range (the set of possible outputs).
- Example:**
- We can define the function square which takes as input an integer x , and outputs its square, x^2 .
 - The domain is set of integers, and the range is the set of positive integers.

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Propositional Function

- A propositional function is a function which
 - Takes one or more arguments.
 - Expresses a predicate involving the argument(s).
 - Becomes a proposition when values are assigned to the arguments.
- The domain of a propositional function is the set of possible values of the arguments.
- **Note:** The domain must be defined so that the value of the propositional function is a proposition for every element of the domain.
- The range of a propositional function is the set of propositions formed by replacing the arguments with each of the possible values from the domain.

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Propositional Function Example

- Let $P(x)$ = “John Cusack was in the movie x .”
- Notice that $P(x)$ is not a proposition, since it is neither true nor false.
- However, $P(\textit{The Matrix})$ = “John Cusack was in the movie *The Matrix*” is a proposition with truth value false.
- In fact, for any movie title x , $P(x)$ is either true or false.
- Therefore, P is a propositional function.
- The domain of P is the set of all movie titles.
(e.g. $P(\textit{laptop})$ is not a proposition.)
- The range of P is the set of all propositions of the form “John Cusack was in the movie x ,” where x is a movie title.

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Universe of Discourse

- We all know that a universe is a particular sphere of interest, activity, or experience.
- You may know that discourse is written or spoken communication.
- So the universe of discourse could be defined loosely as the set of all things we wish to talk about.
- With this in mind, perhaps it is easy to see why the domain of a propositional function is also called the universe of discourse.
- **Example:** What should the universe of discourse be for the propositional function $P(x)$ ="This week the Huskers play against x "

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Multiple Argument Examples

- We mentioned earlier that propositional functions can have more than one argument.

Example 1:

- Let $P(x,y)$ = "x was in the movie y."
- $P(\text{Kevin Bacon}, \text{The Matrix})$ = "Kevin Bacon was in the movie *The Matrix*."
- $P(\text{Kevin Bacon}, \text{The Matrix})$ is a proposition with truth value false.
- What is the universe of discourse for x ? How about y ?
 - For x , the set of all possible actors.
 - For y , the set of all possible movies.

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Multiple Argument Examples

Example 2:

- Let $P(w,x,y,z)$ = " $x^w + y^w = z^w$," where the universe of discourse for w is the set of all positive integers, and for x , y , and z is the set of all integers.
- $P(2,3,4,5)$ is the proposition " $3^2 + 4^2 = 5^2$," which has truth value true.
- For what values of w , x , y , and z , is $P(w,x,y,z)$ false?
(Hint: Have you heard of Fermat's Last Theorem?)

Example 3:

- Let $P(x,y)$ ="x was in a movie with y."
- What should the universe of discourse for x and y be?
- What is the truth value of $P(\text{Kevin Bacon}, \text{Jet Li})$?

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Another Example

Example 4:

- Let $P(x)$ ="x was a movie."
- We can write the statement "*The Matrix* was a movie" as $P(\text{The Matrix})$.
- We can write the statement "*The Matrix* and *E.T.* were movies" as $P(\text{The Matrix}) \wedge P(\text{E.T.})$.
- How should I express a statement like "All things are movies," or "There is at least one movie," or "Not everything is a good movie?"
- What about with notation like "for all x , $P(x)$ is true," or "not for all x , $P(x)$ is true," or "for some x , $P(x)$ is false," or "there is some x for which $P(x)$ is true."
- These sort of do the job, but are too cumbersome.

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Quantifiers

- As the last example demonstrates, there are times when we wish to know whether or not a propositional function is true for
 - all members of the universe of discourse, or
 - no member of the universe of discourse, or
 - any member of the universe of discourse.
- This is where quantifiers come in.
- A quantifier is an expression that indicates the scope of something, specifically in terms of quantity.
- In logic, the two quantifiers we are concerned about are all or every, and some, an, or at least one.

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Universal Quantification

- From our previous definitions of universal and quantifier, hopefully you can guess that the universal quantifier expresses all or every.
- The universal quantification of a propositional function $P(x)$ is the proposition
 - "for all x in the universe of discourse, $P(x)$ is true"
- We can shorten this to "for all x $P(x)$ " or "for every x $P(x)$."
- The notation for this is $\forall x P(x)$.
- The symbol \forall is called the universal quantifier.

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Existential Quantification

- The word existential relates to existence.
- Thus, the existential quantifier expresses some, an, or at least one.
- The existential quantification of a propositional function $P(x)$ is the proposition

“there exists an element x in the universe of discourse such that $P(x)$ is true.”
- We can shorten this to “there exists an x such that $P(x)$,” “for some x $P(x)$,” or “there exists at least one x such that $P(x)$.”
- The notation for this is $\exists x P(x)$.
- The symbol \exists is called the existential quantifier, and is usually pronounced “there exists.”

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Truth Values of Quantifiers

- If the universe of discourse for P is $\{p_1, p_2, \dots, p_n\}$, then

$$\forall x P(x) \Leftrightarrow P(p_1) \wedge P(p_2) \wedge \dots \wedge P(p_n),$$
 and

$$\exists x P(x) \Leftrightarrow P(p_1) \vee P(p_2) \vee \dots \vee P(p_n).$$
- From this, we can easily determine the truth values of the quantifiers.

Truth Values of Quantifiers		
The statement	is true when	is false when
$\forall x P(x)$	$P(x)$ is true for every x	There is at least one x for which $P(x)$ is false
$\exists x P(x)$	There is at least one x for which $P(x)$ is true	$P(x)$ is false for every x

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Quantifier Examples

- Let $P(x)$ = “ x must take a discrete mathematics course,” and $S(x)$ = “ x is a computer science student,” where the universe of discourse for both is all students in a particular college.
- The statement “Every computer science student must take a discrete mathematics course” can be expressed as

$$\forall x (S(x) \rightarrow P(x))$$
- The statement “Everybody must take a discrete mathematics course or be a computer science student” can be expressed as

$$\forall x (S(x) \vee P(x))$$

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More Quantifier Examples

- Let $P(x,y)$ = “ $x+y > 10$ ” where the universe of discourse for x and y is the set of integers.
- The statement “For every x and every y , $x+y > 10$ ” can be expressed as

$$\forall x \forall y P(x,y)$$

- The statement “For every x , there is some value y such that $x+y > 10$ ” can be expressed as

$$\forall x \exists y P(x,y)$$

- The statement “There is a value of y such that for every x , $x+y > 10$ ” can be expressed as

$$\exists y \forall x P(x,y)$$

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Example Continued

- Let $P(x,y)$ = “ $x+y > 10$.”
- What are the truth values of $\forall x \forall y P(x,y)$, $\forall x \exists y P(x,y)$, and $\exists y \forall x P(x,y)$?
- Since $P(1,1)$ is clearly false, $\forall x \forall y P(x,y)$ is also false.
- For any x , if we choose $y = -x + 11$, then

$$P(x,y) = P(x, -x+11) = “x+(-x+11) > 10” = “11 > 10,”$$
 which is clearly true, so $\forall x \exists y P(x,y)$ is true.
- For any y , if $x = -y$, $P(x,y) = “x+(-x) > 10” = “0 > 10,”$ which is false.
- This means we cannot find any y which works with all x , so $\exists y \forall x P(x,y)$ is false.
- What are the truth values if the universe of discourse is the set of positive integers?

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2-Variable Quantifications

- The last example demonstrates that the truth values of $\forall x \exists y P(x,y)$ and $\exists y \forall x P(x,y)$ are not necessarily the same.
- In other words, the order of quantification *can* matter.
- Hopefully it is clear that $\forall x \forall y P(x,y)$ and $\forall y \forall x P(x,y)$ will always have the same truth value, as will $\exists x \exists y P(x,y)$ and $\exists y \exists x P(x,y)$.
- The chart on the next page shows when 2-variable quantifications are true and false.
- We could develop similar charts for quantifications with more variables, but these will suffice for our use.

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Truth Values for 2-Variable Quantifications

Truth Values of 2-Variable Quantifiers		
The statement	is true when	is false when
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x,y .	there is at least one pair x,y for which $P(x,y)$ is false.
$\forall x \exists y P(x,y)$	for every x , there is a y for which $P(x,y)$ is true.	there is an x for which $P(x,y)$ is false for every y .
$\exists x \forall y P(x,y)$	there is an x for which $P(x,y)$ is true for every y .	for every x , there is a y for which $P(x,y)$ is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	there is at least one pair x,y for which $P(x,y)$ is true.	$P(x,y)$ is false for every pair x,y .

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The Unique Parent

Problem:

- Write the English sentence "Everybody has exactly one biological mother" as a logical expression.

Solution:

- Let $P(x,y)$ = "x is the biological mother of y."
- The expression $\forall y \exists x P(x,y)$ looks correct.
- However, this expresses the sentence "Everybody has a biological mother," which doesn't guarantee that somebody doesn't have more than one.
- We need to express that a person can have at most one.
- The correct solution is

$$\forall y \exists x \forall z (P(x,y) \wedge [(z \neq x) \rightarrow \neg P(z,y)])$$

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Mathematical Laws

Problems:

- What does the logical expression $\forall x \forall y (x+y = y+x)$ mean, where the universe of discourse for x and y is the set of real numbers? What is its truth value?
- What does the logical expression $\forall x \exists y (x*y = 1)$ mean, where the universe of discourse for x and y is the set of rational numbers? What is its truth value?

Solutions:

- This is simply the commutative law of addition for real numbers, which is true.
- This is the multiplicative inverse law for rational numbers, which is true.

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More Mathematical Laws?

Problems:

- What does the logical expression $\forall x \forall y (x-y = y-x)$ mean, where the universe of discourse for x and y is the set of real numbers? What is its truth value?
- What does the logical expression $\forall x \exists y (x*y = 1)$ mean, where the universe of discourse for x and y is the set of integers? What is its truth value?

Solutions:

- This would be the commutative law of subtraction for real numbers, but it doesn't exist, since it is false.
- This would be the multiplicative inverse law for integers, but there is no such law, since it is false.

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A Special Number

Problem:

- Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x " as a logical expression.

Solution:

- Let $P(x,y) = "x+y = y,"$ and $Q(x,y) = "x*y = x."$
- Then the solution is

$$\exists x \forall y (P(x,y) \wedge Q(x,y))$$

Which we can also write as

$$\exists x \forall y ((x+y = y) \wedge (x*y = x))$$

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And the Number is ...

Problem:

- What is the truth value of $\exists x \forall y ((x+y = y) \wedge (x*y = x))$ if the universe of discourse for x and y is the set of integers? Real numbers? Rational numbers? Positive integers?
- If the truth value is true, what is the value x ?

Solution:

- For the set of integers, real numbers, and rational numbers, the expression is true, with $x=0$.
- For the set of positive integers, the truth value is false.

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Negations and Quantifiers I

Problem:

- Express the statement “Not everybody can ride a bike” as a logical expression.

Solution:

- Let $P(x)$ = “ x can ride a bike.”
- The statement “everybody can ride a bike,” can be expressed as $\forall x P(x)$.
- We want the negation of this, which is $\neg \forall x P(x)$.
- Another way to say this is “There is somebody that cannot ride a bike,” which can be expressed as $\exists x \neg P(x)$.

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Negations and Quantifiers II

Problem:

- Express the statement “Nobody can fly.” as a logical expression.

Solution:

- Let $P(x)$ = “ x can fly.”
- The statement “somebody can fly,” can be expressed as $\exists x P(x)$.
- We want the negation of this, which is $\neg \exists x P(x)$.
- Another way to say this is “Everybody can not fly,” which can be expressed as $\forall x \neg P(x)$.

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Negating Quantifiers: The Rules

- In the last two examples, we were able to express each statement in two different ways.
- In fact, for *any* propositional function, this is the case.
- The following chart shows the rules.

Negating Quantifiers			
Negation	Equivalent	Is true when	Is false when
$\neg \forall x P(x)$	$\exists x \neg P(x)$	there is an x for which $P(x)$ is false.	$P(x)$ is true for every x .
$\neg \exists x P(x)$	$\forall x \neg P(x)$	$P(x)$ is false for every x .	there is an x for which $P(x)$ is true.

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Exercises

- What is the truth value of $\exists x \forall y (x * y = y)$ if the universe of discourse for x and y is the set of real numbers? What does this expression mean?
- Let $Q(x,y)$ be the statement “ x is the capital city of y .” What should the universe of discourse be for x and y ? What are the truth values of $\forall x \exists y Q(x,y)$ and $\exists x \forall y Q(x,y)$?
- If $P(x)$ = “ $x < 0$,” what are the truth values of $P(2)$, $P(0)$, and $P(-1)$?
- Express the statement “everybody in this class is a student in this university” as a logical expression.
- Express the statement “Either somebody has taken every class, or everybody has taken exactly one class” as a logical expression.
- Let A be an array of size n , and $P(i)$ = “ $A[i] = 0$.” You need to implement the following:

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if (  $\neg \forall x P(x)$  ) Initialize all elements of  $A$  to 0
Give C++ or Java code to implement this. What does this code do?

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More Exercises

- Let $P(x,y)$ = “ x is a factor of y ,” where the universe of discourse for x is $\{1,2,3,\dots\}$ and for y is $\{2,3,4,\dots\}$.
 Let $Q(y)$ = “ $\forall x (P(x,y) \rightarrow (x = y) \vee (x = 1))$.”
 When is $Q(y)$ true? Give an alternative way of writing $Q(y)$.
- Let $P(x,y)$ = “ $x^2 - y^2 = 0$,” where the universe of discourse for x and y is the set of integers. What are the truth values of $P(4,-4)$, $P(0,3)$, $\forall x \forall y P(x,y)$, $\forall x \exists y P(x,y)$, $\exists x \exists y P(x,y)$, and $\exists y \forall x P(x,y)$?
- Rewrite the expression $\neg \forall x (\exists y \forall z P(x,y,z) \wedge \exists z \forall y P(x,y,z))$ so no negations appear to the left of a quantifier.
- You need to create a random number generator that inputs an integer from 1 to 1,000,000 as a random seed, and outputs a random number from 1 to 1,000. It must be possible for each number between 1 and 1,000 to be generated. Express the requirements for this function using quantifier(s) and propositional function(s).

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