CSCE 230J
Computer Organization

Floating Point

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Giving credit where credit is due

- **Most of slides for this lecture are based on slides created by Drs. Bryant and O'Hallaron, Carnegie Mellon University.**
- **I have modified them and added new slides.**

Topics

- **IEEE Floating Point Standard**
- **Rounding**
- **Floating Point Operations**
- **Mathematical properties**

Floating Point Puzzles

- **For each of the following C expressions, either:**
  - **Argue that it is true for all argument values**
  - **Explain why not true**

```
int x = …;
float f = …;
double d = …;
```

Assume neither d nor f is NaN

```
• x == (int)(float) x
• x == (int)(double) x
• f == (float)(double) f
• d == (float) d
• f == -(-f);
• 2/3 == 2/3.0
• d < 0.0     ⇒  ((d*2) < 0.0)
• d > f       ⇒  -f < -d
• d * d >= 0.0
• (d+f)-d == f
```

IEEE Floating Point

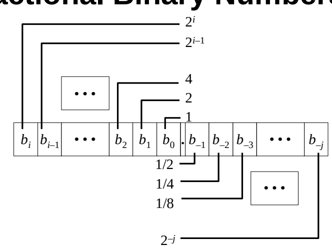
IEEE Standard 754

- **Established in 1985 as uniform standard for floating point arithmetic**
  - **Before that, many idiosyncratic formats**
- **Supported by all major CPUs**

Driven by Numerical Concerns

- **Nice standards for rounding, overflow, underflow**
- **Hard to make go fast**
  - **Numerical analysts predominated over hardware types in defining standard**

Fractional Binary Numbers



Representation

- **Bits to right of "binary point" represent fractional powers of 2**
- **Represents rational number:** $\sum_{k=-j}^{i} b_k \cdot 2^k$

Frac. Binary Number Examples

Value	Representation
5-3/4	`101.11`$_2$
2-7/8	`10.111`$_2$
63/64	`0.111111`$_2$

Observations

- Divide by 2 by shifting right
- Multiply by 2 by shifting left
- Numbers of form `0.111111…`$_2$ just below 1.0
  - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
  - Use notation $1.0 - \varepsilon$

Representable Numbers

Limitation

- Can only exactly represent numbers of the form $x/2^k$
- Other numbers have repeating bit representations

Value	Representation
1/3	`0.0101010101[01]…`$_2$
1/5	`0.001100110011[0011]…`$_2$
1/10	`0.0001100110011[0011]…`$_2$

Floating Point Representation

Numerical Form

- $-1^s\ M\ 2^E$
  - Sign bit *s* determines whether number is negative or positive
  - Significand *M* normally a fractional value in range [1.0,2.0).
  - Exponent *E* weights value by power of two

Encoding

s	exp	frac

- MSB is sign bit
- `exp` field encodes *E*
- `frac` field encodes *M*

Floating Point Precisions

Encoding

s	exp	frac

- MSB is sign bit
- `exp` field encodes *E*
- `frac` field encodes *M*

Sizes

- Single precision: 8 `exp` bits, 23 `frac` bits
  - 32 bits total
- Double precision: 11 `exp` bits, 52 `frac` bits
  - 64 bits total
- Extended precision: 15 `exp` bits, 63 `frac` bits
  - Only found in Intel-compatible machines
  - Stored in 80 bits
    - » 1 bit wasted

"Normalized" Numeric Values

Condition

- `exp` ≠ `000…0` and `exp` ≠ `111…1`

Exponent coded as *biased* value

E = Exp – Bias

- *Exp* : unsigned value denoted by exp
- *Bias* : Bias value
  - » Single precision: 127 (*Exp*: 1…254, *E*: -126…127)
  - » Double precision: 1023 (*Exp*: 1…2046, *E*: -1022…1023)
  - » in general: *Bias* = 2^{e-1} - 1, where e is number of exponent bits

Significand coded with implied leading 1

M = `1.xxx…x`$_2$

- `xxx…x`: bits of `frac`
- Minimum when `000…0` (*M* = 1.0)
- Maximum when `111…1` (*M* = $2.0 - \varepsilon$)
- Get extra leading bit for "free"

Normalized Encoding Example

Value

`Float F = 15213.0;`

- $15213_{10} = 11101101101101_2 = 1.1101101101101_2 \times 2^{13}$

Significand

M = $1.\underline{1101101101101}_2$

`frac` = $\underline{1101101101101}0000000000_2$

Exponent

E = 13

Bias = 127

Exp = 140 = 10001100_2

Floating Point Representation (Class 02):

Hex:	4	6	6	D	B	4	0	0
Binary:	0100	0110	0110	1101	1011	0100	0000	0000

140: `100 0110 0`

15213: `1110 1101 1011 01`

Denormalized Values

Condition

- `exp` = `000…0`

Value

- Exponent value *E* = –*Bias* + 1
- Significand value *M* = `0.xxx…x`$_2$
  - `xxx…x`: bits of `frac`

Cases

- `exp` = `000…0`, `frac` = `000…0`
  - Represents value 0
  - Note that have distinct values +0 and –0
- `exp` = `000…0`, `frac` ≠ `000…0`
  - Numbers very close to 0.0
  - Lose precision as get smaller
  - "Gradual underflow"

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Special Values

Condition

- `exp` = `111…1`

Cases

- `exp` = `111…1`, `frac` = `000…0`
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., 1.0/0.0 = –1.0/–0.0 = +∞, 1.0/–0.0 = –∞
- `exp` = `111…1`, `frac` ≠ `000…0`
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), ∞ – ∞

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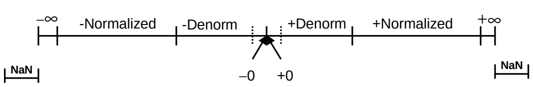
Summary of Floating Point Real Number Encodings



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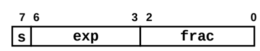
Tiny Floating Point Example

8-bit Floating Point Representation

- the sign bit is in the most significant bit.
- the next four bits are the exponent, with a bias of 7.
- the last three bits are the `frac`

- **Same General Form as IEEE Format**
  - normalized, denormalized
  - representation of 0, NaN, infinity



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Values Related to the Exponent

Exp	exp	E	2^E	
0	0000	-6	1/64	(denorms)
1	0001	-6	1/64	
2	0010	-5	1/32	
3	0011	-4	1/16	
4	0100	-3	1/8	
5	0101	-2	1/4	
6	0110	-1	1/2	
7	0111	0	1	
8	1000	+1	2	
9	1001	+2	4	
10	1010	+3	8	
11	1011	+4	16	
12	1100	+5	32	
13	1101	+6	64	
14	1110	+7	128	
15	1111	n/a		(inf, NaN)

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Dynamic Range

	s exp frac	*E*	Value	
Denormalized numbers	0 0000 000	-6	0	
	0 0000 001	-6	1/8*1/64 = 1/512	← closest to zero
	0 0000 010	-6	2/8*1/64 = 2/512	
	…			
	0 0000 110	-6	6/8*1/64 = 6/512	
	0 0000 111	-6	7/8*1/64 = 7/512	← largest denorm
Normalized numbers	0 0001 000	-6	8/8*1/64 = 8/512	← smallest norm
	0 0001 001	-6	9/8*1/64 = 9/512	
	…			
	0 0110 110	-1	14/8*1/2 = 14/16	
	0 0110 111	-1	15/8*1/2 = 15/16	← closest to 1 below
	0 0111 000	0	8/8*1 = 1	
	0 0111 001	0	9/8*1 = 9/8	← closest to 1 above
	0 0111 010	0	10/8*1 = 10/8	
	…			
	0 1110 110	7	14/8*128 = 224	
	0 1110 111	7	15/8*128 = 240	← largest norm
	0 1111 000	n/a	inf	

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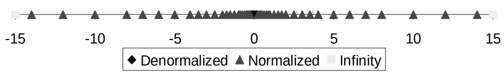
Distribution of Values

6-bit IEEE-like format

- **e = 3 exponent bits**
- **f = 2 fraction bits**
- **Bias is 3**

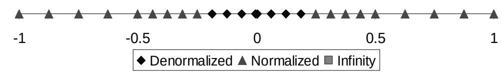
Notice how the distribution gets denser toward zero.



Distribution of Values (close-up view)

6-bit IEEE-like format

- **e = 3 exponent bits**
- **f = 2 fraction bits**
- **Bias is 3**



Interesting Numbers

Description	**`exp`**	**`frac`**	**Numeric Value**
Zero	00…00	00…00	0.0
Smallest Pos. Denorm.	00…00	00…01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
■ Single ≈ 1.4×10^{-45}			
■ Double ≈ 4.9×10^{-324}			
Largest Denormalized	00…00	11…11	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
■ Single ≈ 1.18×10^{-38}			
■ Double ≈ 2.2×10^{-308}			
Smallest Pos. Normalized	00…01	00…00	$1.0 \times 2^{-\{126,1022\}}$
■ Just larger than largest denormalized			
One	01…11	00…00	1.0
Largest Normalized	11…10	11…11	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$
■ Single ≈ 3.4×10^{38}			
■ Double ≈ 1.8×10^{308}			

Special Properties of Encoding

FP Zero Same as Integer Zero

- **All bits = 0**

Can (Almost) Use Unsigned Integer Comparison

- **Must first compare sign bits**
- **Must consider -0 = 0**
- **NaNs problematic**
  - **Will be greater than any other values**
  - **What should comparison yield?**
- **Otherwise OK**
  - **Denorm vs. normalized**
  - **Normalized vs. infinity**

Floating Point Operations

Conceptual View

- **First compute exact result**
- **Make it fit into desired precision**
  - **Possibly overflow if exponent too large**
  - **Possibly round to fit into `frac`**

Rounding Modes (illustrate with $ rounding)

	$1.40	**$1.60**	**$1.50**	**$2.50**	**–$1.50**
■ **Zero**	$1	$1	$1	$2	–$1
■ **Round down (-∞)**	$1	$1	$1	$2	–$2
■ **Round up (+∞)**	$2	$2	$2	$3	–$1
■ **Nearest Even (default)**	$1	$2	$2	$2	–$2

Note:

1. **Round down: rounded result is close to but no greater than true result.**
2. **Round up: rounded result is close to but no less than true result.**

Closer Look at Round-To-Even

Default Rounding Mode

- **Hard to get any other kind without dropping into assembly**
- **All others are statistically biased**
  - **Sum of set of positive numbers will consistently be over- or under-estimated**

Applying to Other Decimal Places / Bit Positions

- **When exactly halfway between two possible values**
  - **Round so that least significant digit is even**
- **E.g., round to nearest hundredth**

1.2349999	**1.23**	**(Less than half way)**
1.2350001	**1.24**	**(Greater than half way)**
1.2350000	**1.24**	**(Half way—round up)**
1.2450000	**1.24**	**(Half way—round down)**

Rounding Binary Numbers

Binary Fractional Numbers

- "Even" when least significant bit is `0`
- Half way when bits to right of rounding position = `100…`$_2$

Examples

- Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

FP Multiplication

Operands

$(-1)^{s1}\ M1\ \ 2^{E1}\quad * \quad (-1)^{s2}\ M2\ \ 2^{E2}$

Exact Result

$(-1)^{s}\ M\ \ 2^{E}$

- **Sign** *s*: $s1 \wedge s2$
- **Significand** *M*: $M1 * M2$
- **Exponent** *E*: $E1 + E2$

Fixing

- If $M \geq 2$, shift *M* right, increment *E*
- If *E* out of range, overflow
- Round *M* to fit `frac` precision

Implementation

- Biggest chore is multiplying significands

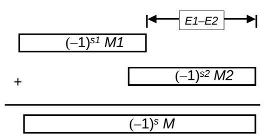
FP Addition

Operands

$(-1)^{s1}\ M1\ \ 2^{E1}$

$(-1)^{s2}\ M2\ \ 2^{E2}$

- Assume $E1 > E2$



Exact Result

$(-1)^{s}\ M\ \ 2^{E}$

- Sign *s*, significand *M*:
  - Result of signed align & add
- Exponent *E*: $E1$

Fixing

- If $M \geq 2$, shift *M* right, increment *E*
- if $M < 1$, shift *M* left *k* positions, decrement *E* by *k*
- Overflow if *E* out of range
- Round *M* to fit `frac` precision

Mathematical Properties of FP Add

Compare to those of Abelian Group

- Closed under addition? **YES**
  - But may generate infinity or NaN
- Commutative? **YES**
- Associative? **NO**
  - Overflow and inexactness of rounding
- 0 is additive identity? **YES**
- Every element has additive inverse **ALMOST**
  - Except for infinities & NaNs

Monotonicity

- $a \geq b \Rightarrow a+c \geq b+c$? **ALMOST**
  - Except for infinities & NaNs

Math. Properties of FP Mult

Compare to Commutative Ring

- Closed under multiplication? **YES**
  - But may generate infinity or NaN
- Multiplication Commutative? **YES**
- Multiplication is Associative? **NO**
  - Possibility of overflow, inexactness of rounding
- 1 is multiplicative identity? **YES**
- Multiplication distributes over addition? **NO**
  - Possibility of overflow, inexactness of rounding

Monotonicity

- $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$? **ALMOST**
  - Except for infinities & NaNs

Floating Point in C

C Guarantees Two Levels

`float` single precision

`double` double precision

Conversions

- Casting between `int`, `float`, and `double` changes numeric values
- `Double` or `float` to `int`
  - Truncates fractional part
  - Like rounding toward zero
  - Not defined when out of range
    - » Generally saturates to TMin or TMax
- `int` to `double`
  - Exact conversion, as long as int has ≤ 53 bit word size
- `int` to `float`
  - Will round according to rounding mode

Answers to Floating Point Puzzles

```
int x = …;
float f = …;
double d = …;
```

Assume neither d nor f is NAN

`x == (int)(float) x`	No: 24 bit significand
`x == (int)(double) x`	Yes: 53 bit significand
`f == (float)(double) f`	Yes: increases precision
`d == (float) d`	No: loses precision
`f == -(-f);`	Yes: Just change sign bit
`2/3 == 2/3.0`	No: 2/3 == 0
`d < 0.0 ⇒((d*2) < 0.0)`	Yes!
`d > f ⇒-f < -d`	Yes!
`d * d >= 0.0`	Yes!
`(d+f)-d == f`	No: Not associative

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Ariane 5

- Exploded 37 seconds after liftoff
- Cargo worth $500 million

Why

- Computed horizontal velocity as floating point number
- Converted to 16-bit integer
- Worked OK for Ariane 4
- Overflowed for Ariane 5
  - Used same software

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Summary

IEEE Floating Point Has Clear Mathematical Properties

- Represents numbers of form $M \times 2^E$
- Can reason about operations independent of implementation
  - As if computed with perfect precision and then rounded
- Not the same as real arithmetic
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers

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