CSCE 230J
Computer Organization

Integers

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Giving credit where credit is due

- **Most of slides for this lecture are based on slides created by Drs. Bryant and O'Hallaron, Carnegie Mellon University.**
- **I have modified them and added new slides.**

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Topics

- Numeric Encodings
  - Unsigned & Two's complement
- Programming Implications
  - C promotion rules
- Basic operations
  - Addition, negation, multiplication
- Programming Implications
  - Consequences of overflow
  - Using shifts to perform power-of-2 multiply/divide

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C Puzzles

- Taken from old exams
- Assume machine with 32 bit word size, two's complement integers
- For each of the following C expressions, either:
  - Argue that is true for all argument values
  - Give example where not true

Initialization

```
int x = foo();
int y = bar();
unsigned ux = x;
unsigned uy = y;
```

- `x < 0` ⇒ `((x*2) < 0)`
- `ux >= 0`
- `x & 7 == 7` ⇒ `(x<<30) < 0`
- `ux > -1`
- `x > y` ⇒ `-x < -y`
- `x * x >= 0`
- `x > 0 && y > 0` ⇒ `x + y > 0`
- `x >= 0` ⇒ `-x <= 0`
- `x <= 0` ⇒ `-x >= 0`

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Encoding Integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

Sign Bit

```
short int x =  15213;
short int y = -15213;
```

- **C `short` 2 bytes long**

	Decimal	Hex	Binary
x	15213	`3B 6D`	`00111011 01101101`
y	-15213	`C4 93`	`11000100 10010011`

Sign Bit

- **For 2's complement, most significant bit indicates sign**
  - **0 for nonnegative**
  - **1 for negative**

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Encoding Example (Cont.)

```
x =      15213: 00111011 01101101
y =     -15213: 11000100 10010011
```

Weight	15213		-15213	
1	1	1	1	1
2	0	0	1	2
4	1	4	0	0
8	1	8	0	0
16	0	0	1	16
32	1	32	0	0
64	1	64	0	0
128	0	0	1	128
256	1	256	0	0
512	1	512	0	0
1024	0	0	1	1024
2048	1	2048	0	0
4096	1	4096	0	0
8192	1	8192	0	0
16384	0	0	1	16384
-32768	0	0	1	-32768
Sum		**15213**		**-15213**

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Numeric Ranges

Unsigned Values

- *UMin* = 0
  000…0
- *UMax* = $2^w - 1$
  111…1

Two's Complement Values

- *TMin* = -2^{w-1}
  100…0
- *TMax* = $2^{w-1} - 1$
  011…1

Other Values

- Minus 1
  111…1

Values for *W* = 16

	Decimal	Hex	Binary
`UMax`	65535	`FF FF`	`11111111 11111111`
`TMax`	32767	`7F FF`	`01111111 11111111`
`TMin`	-32768	`80 00`	`10000000 00000000`
`-1`	-1	`FF FF`	`11111111 11111111`
`0`	0	`00 00`	`00000000 00000000`

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Values for Different Word Sizes

	W			
	8	**16**	**32**	**64**
UMax	**255**	**65,535**	**4,294,967,295**	**18,446,744,073,709,551,615**
TMax	**127**	**32,767**	**2,147,483,647**	**9,223,372,036,854,775,807**
TMin	**-128**	**-32,768**	**-2,147,483,648**	**-9,223,372,036,854,775,808**

Observations

- |*TMin* | = *TMax* + 1
  - Asymmetric range
- *UMax* = 2 * *TMax* + 1

C Programming

- **`#include <limits.h>`**
  - **K&R App. B11**
- **Declares constants, e.g.,**
  - **`ULONG_MAX`**
  - **`LONG_MAX`**
  - **`LONG_MIN`**
- **Values platform-specific**

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Unsigned & Signed Numeric Values

X	B2U(*X*)	B2T(*X*)
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

Equivalence

- **Same encodings for nonnegative values**

Uniqueness

- **Every bit pattern represents unique integer value**
- **Each representable integer has unique bit encoding**

$\Rightarrow$ Can Invert Mappings

- **U2B(*x*) = B2U^{-1}(*x*)**
  - **Bit pattern for unsigned integer**
- **T2B(*x*) = B2T^{-1}(*x*)**
  - **Bit pattern for two's comp integer**

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Casting Signed to Unsigned

C Allows Conversions from Signed to Unsigned

```
short int          x =  15213;
unsigned short int ux = (unsigned short) x;
short int          y  = -15213;
unsigned short int uy = (unsigned short) y;
```

Resulting Value

- **No change in bit representation**
- **Nonnegative values unchanged**
  - ***ux* = 15213**
- **Negative values change into (large) positive values**
  - ***uy* = 50323**

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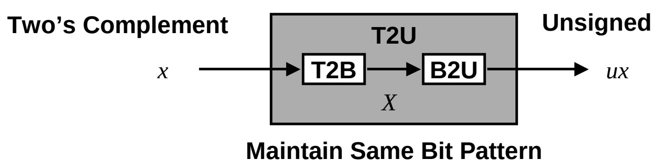
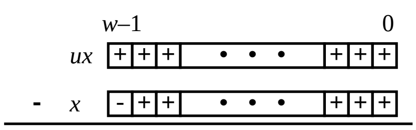
Relation between Signed & Unsigned





$+2^{w-1} - -2^{w-1} = 2*2^{w-1} = 2^w$

$$ux = \begin{cases} x & x \geq 0 \\ x + 2^w & x < 0 \end{cases}$$

Relation Between Signed & Unsigned

Weight	-15213		50323	
1	1	1	1	1
2	1	2	1	2
4	0	0	0	0
8	0	0	0	0
16	1	16	1	16
32	0	0	0	0
64	0	0	0	0
128	1	128	1	128
256	0	0	0	0
512	0	0	0	0
1024	1	1024	1	1024
2048	0	0	0	0
4096	0	0	0	0
8192	0	0	0	0
16384	1	16384	1	16384
32768	1	*-32768*	1	*32768*
Sum		**-15213**		**50323**

- $uy = y + 2 * 32768 = y + 65536$

Signed vs. Unsigned in C

Constants

- By default are considered to be signed integers
- Unsigned if have "U" as suffix

```
0U, 4294967259U
```

Casting

- Explicit casting between signed & unsigned same as U2T and T2U

```
int tx, ty;
unsigned ux, uy;
tx = (int) ux;
uy = (unsigned) ty;
```

- Implicit casting also occurs via assignments and procedure calls

```
tx = ux;
uy = ty;
```

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Casting Surprises

Expression Evaluation

- If mix unsigned and signed in single expression, signed values implicitly cast to unsigned
- Including comparison operations <, >, ==, <=, >=
- Examples for *W* = 32

Constant$_1$	Constant$_2$	Relation	Evaluation
`0`	`0U`	==	unsigned
`-1`	`0`	<	signed
`-1`	`0U`	>	unsigned
`2147483647`	`-2147483648`	>	signed
`2147483647U`	`-2147483648`	<	unsigned
`-1`	`-2`	>	signed
`(unsigned) -1`	`-2`	>	unsigned
`2147483647`	`2147483648U`	<	unsigned
`2147483647`	`(int) 2147483648U`	>	signed

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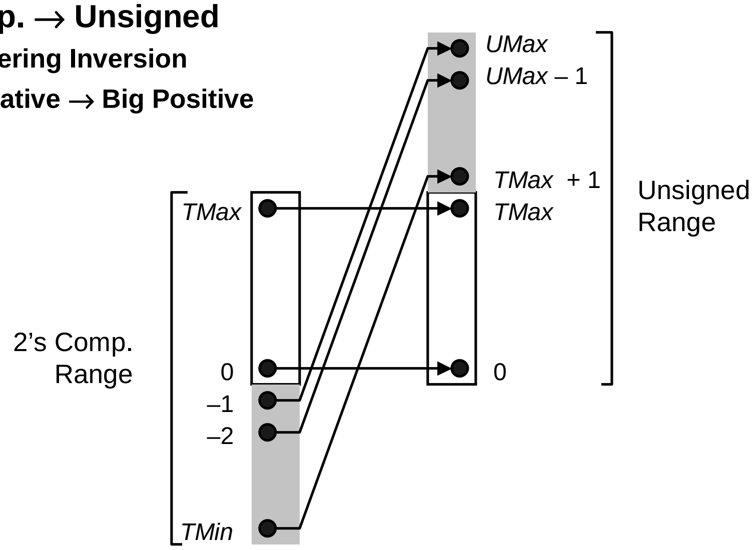
Explanation of Casting Surprises

2's Comp. → Unsigned

- **Ordering Inversion**
- **Negative → Big Positive**



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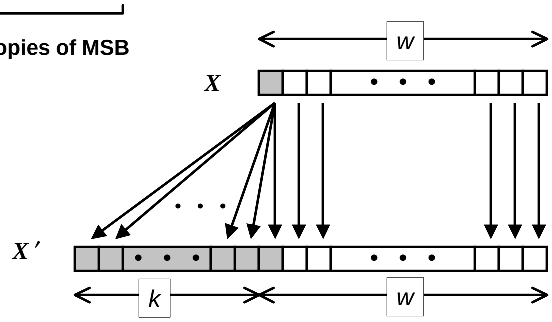
Sign Extension

Task:

- **Given *w*-bit signed integer *x***
- **Convert it to *w*+*k*-bit integer with same value**

Rule:

- **Make *k* copies of sign bit:**
- $X' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2}, \ldots, x_0$



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Sign Extension Example

```
short int x =  15213;
int      ix = (int) x;
short int y = -15213;
int      iy = (int) y;
```

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- **Converting from smaller to larger integer data type**
- **C automatically performs sign extension**

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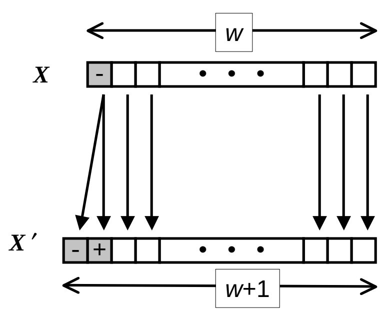
Justification For Sign Extension

Prove Correctness by Induction on *k*

- **Induction Step: extending by single bit maintains value**



- **Key observation:** $-2^{w-1} = -2^{w} + 2^{w-1}$
- **Look at weight of upper bits:**

$x \quad -2^{w-1}\, x_{w-1}$

$x' \quad -2^{w}\, x_{w-1} + 2^{w-1}\, x_{w-1} \quad = \quad -2^{w-1}\, x_{w-1}$

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Why Should I Use Unsigned?

Don't **Use Just Because Number Nonzero**

- C compilers on some machines generate less efficient code

```
unsigned i;
for (i = 1; i < cnt; i++)
  a[i] += a[i-1];
```

- Easy to make mistakes

```
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

Do **Use When Performing Modular Arithmetic**

- Multiprecision arithmetic
- Other esoteric stuff

Do **Use When Need Extra Bit's Worth of Range**

- Working right up to limit of word size

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Negating with Complement & Increment

Claim: Following Holds for 2's Complement

`~x + 1 == -x`

Complement

- Observation: `~x + x == 1111…11`$_2$ `== -1`

`x`	1	0	0	1	1	1	0	1
`+ ~x`	0	1	1	0	0	0	1	0
`-1`	1	1	1	1	1	1	1	1

Increment

- $\sim x + \cancel{x} + (\cancel{-x} + 1) \quad == \cancel{-1} + (-x + \cancel{1})$
- `~x + 1      ==    -x`

Warning: Be cautious treating `int`'s as integers

- OK here

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Comp. & Incr. Examples

x = 15213

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
~x	-15214	C4 92	11000100 10010010
~x+1	-15213	C4 93	11000100 1001001**1**
y	-15213	C4 93	11000100 10010011

0

	Decimal	Hex	Binary
0	0	00 00	00000000 00000000
~0	-1	FF FF	11111111 11111111
~0+1	0	00 00	00000000 00000000

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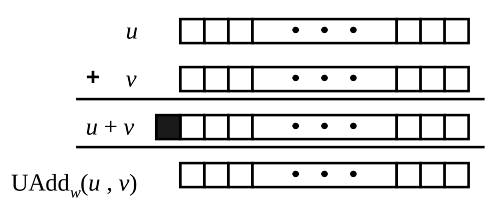
Unsigned Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



Standard Addition Function

- **Ignores carry output**

Implements Modular Arithmetic

$s = \mathrm{UAdd}_w(u, v) = u + v \bmod 2^w$

$$UAdd_w(u,v) = \begin{cases} u+v & u+v < 2^w \\ u+v-2^w & u+v \geq 2^w \end{cases}$$

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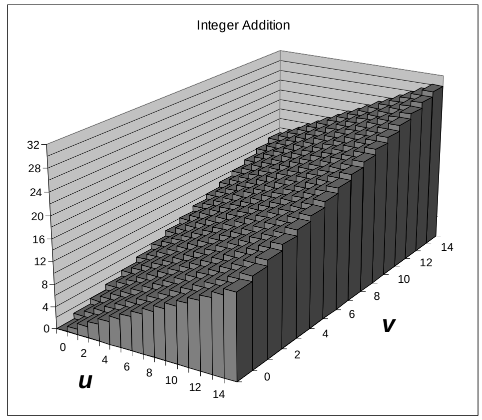
Visualizing Integer Addition

Integer Addition

- **4-bit integers *u*, *v***
- **Compute true sum $\text{Add}_4(u, v)$**
- **Values increase linearly with *u* and *v***
- **Forms planar surface**

$\text{Add}_4(u , v)$



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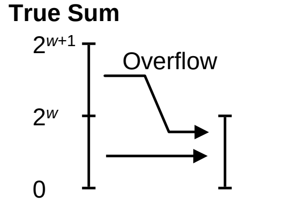
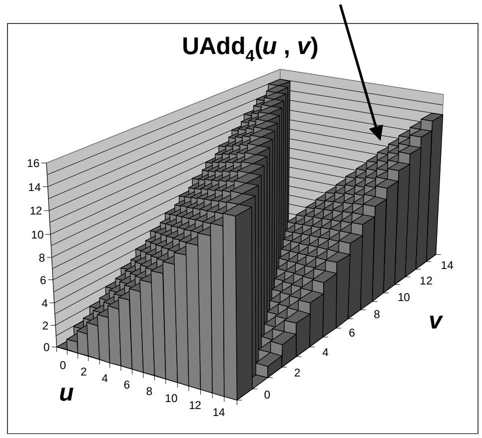
Visualizing Unsigned Addition

Wraps Around

- **If true sum ≥ 2^w**
- **At most once**



Overflow



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Mathematical Properties

Modular Addition Forms an *Abelian Group*

- **Closed under addition**
  $0 \le \mathrm{UAdd}_w(u, v) \le 2^w - 1$
- **Commutative**
  $\mathrm{UAdd}_w(u, v) = \mathrm{UAdd}_w(v, u)$
- **Associative**
  $\mathrm{UAdd}_w(t, \mathrm{UAdd}_w(u, v)) = \mathrm{UAdd}_w(\mathrm{UAdd}_w(t, u), v)$
- **0 is additive identity**
  $\mathrm{UAdd}_w(u, 0) = u$
- **Every element has additive inverse**
  - Let $\mathrm{UComp}_w(u) = 2^w - u$

  $\mathrm{UAdd}_w(u, \mathrm{UComp}_w(u)) = 0$

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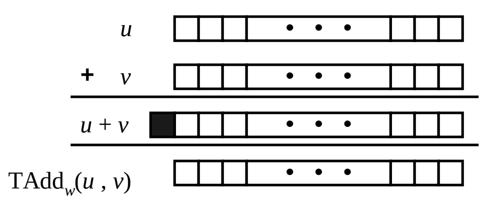
Two's Complement Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



TAdd and UAdd have Identical Bit-Level Behavior

- **Signed vs. unsigned addition in C:**

```
int s, t, u, v;
s = (int) ((unsigned) u + (unsigned) v);
 t = u + v
```

- **Will give** `s == t`

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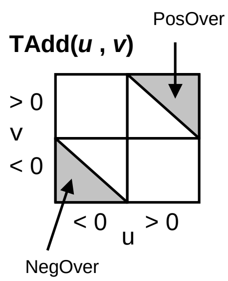
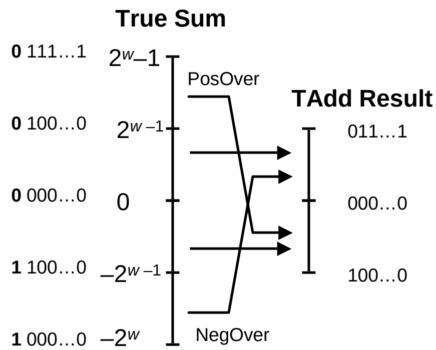
Characterizing TAdd

Functionality

- **True sum requires** *w*+1 **bits**
- **Drop off MSB**
- **Treat remaining bits as 2's comp. integer**





$$TAdd_w(u,v) = \begin{cases} u+v+2^{w-1} & u+v < TMin_w \quad \textbf{(NegOver)} \\ u+v & TMin_w \le u+v \le TMax_w \\ u+v-2^{w-1} & TMax_w < u+v \quad \textbf{(PosOver)} \end{cases}$$

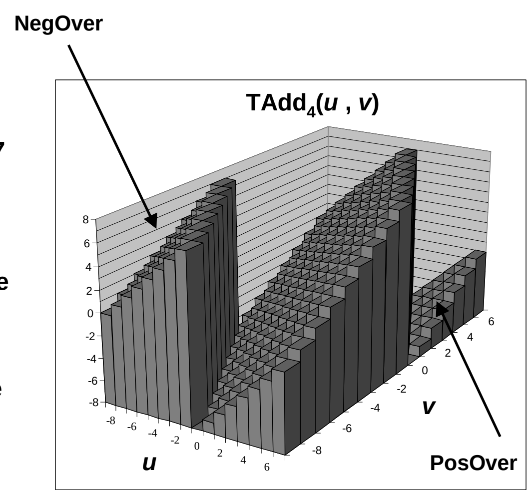
Visualizing 2's Comp. Addition

Values

- **4-bit two's comp.**
- **Range from -8 to +7**

Wraps Around

- **If sum ≥ 2^{w-1}**
  - **Becomes negative**
  - **At most once**
- **If sum < -2^{w-1}**
  - **Becomes positive**
  - **At most once**

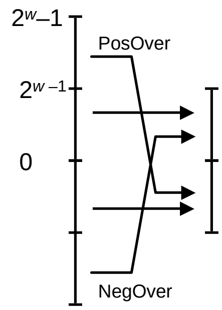


Detecting 2's Comp. Overflow

Task

- **Given** $s = \text{TAdd}_w(u, v)$
- **Determine if** $s = \text{Add}_w(u, v)$
- **Example**

```
int s, u, v;
s = u + v;
```



Claim

- **Overflow iff either:**
  - $u, v < 0, s \geq 0$ **(NegOver)**
  - $u, v \geq 0, s < 0$ **(PosOver)**

```
ovf = (u<0 == v<0) && (u<0 != s<0);
```

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Mathematical Properties of TAdd

Isomorphic Algebra to UAdd

- $\text{TAdd}_w(u, v) = \text{U2T}(\text{UAdd}_w(\text{T2U}(u), \text{T2U}(v)))$
  - **Since both have identical bit patterns**

Two's Complement Under TAdd Forms a Group

- **Closed, Commutative, Associative, 0 is additive identity**
- **Every element has additive inverse**

Let $\text{TComp}_w(u) = \text{U2T}(\text{UComp}_w(\text{T2U}(u)))$

$\text{TAdd}_w(u, \text{TComp}_w(u)) = 0$

$$TComp_w(u) = \begin{cases} -u & u \neq TMin_w \\ TMin_w & u = TMin_w \end{cases}$$

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Multiplication

Computing Exact Product of *w*-bit numbers *x*, *y*

- **Either signed or unsigned**

Ranges

- **Unsigned:** $0 \le x * y \le (2^w - 1)^2 = 2^{2w} - 2^{w+1} + 1$
  - **Up to 2*w* bits**
- **Two's complement min:** $x * y \ge (-2^{w-1})*(2^{w-1}-1) = -2^{2w-2} + 2^{w-1}$
  - **Up to 2*w*–1 bits**
- **Two's complement max:** $x * y \le (-2^{w-1})^2 = 2^{2w-2}$
  - **Up to 2*w* bits, but only for** $(TMin_w)^2$

Maintaining Exact Results

- **Would need to keep expanding word size with each product computed**
- **Done in software by "arbitrary precision" arithmetic packages**

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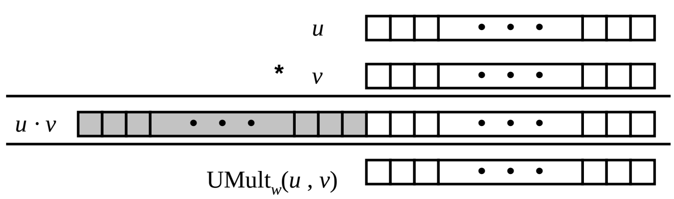
Unsigned Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits



Standard Multiplication Function

- **Ignores high order *w* bits**

Implements Modular Arithmetic

$\mathrm{UMult}_w(u, v) = u \cdot v \bmod 2^w$

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Unsigned vs. Signed Multiplication

Unsigned Multiplication

```
unsigned ux = (unsigned) x;
unsigned uy = (unsigned) y;
unsigned up = ux * uy
```

- **Truncates product to *w*-bit number** *up* **=** $\text{UMult}_w(ux, uy)$
- **Modular arithmetic:** $up = ux \cdot uy \mod 2^w$

Two's Complement Multiplication

```
int x, y;
int p = x * y;
```

- **Compute exact product of two *w*-bit numbers *x*, *y***
- **Truncate result to *w*-bit number** *p* $= \text{TMult}_w(x, y)$

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Unsigned vs. Signed Multiplication

Unsigned Multiplication

```
unsigned ux = (unsigned) x;
unsigned uy = (unsigned) y;
unsigned up = ux * uy
```

Two's Complement Multiplication

```
int x, y;
int p = x * y;
```

Relation

- **Signed multiplication gives same bit-level result as unsigned**
- `up == (unsigned) p`

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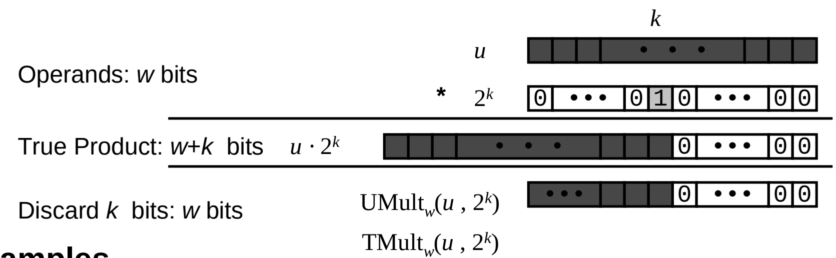
Power-of-2 Multiply with Shift

Operation

- **u << k gives u * 2^k**
- **Both signed and unsigned**



Examples

- **`u << 3 == u * 8`**
- **`u << 5 - u << 3 == u * 24`**
- **Most machines shift and add much faster than multiply**
  - **Compiler generates this code automatically**

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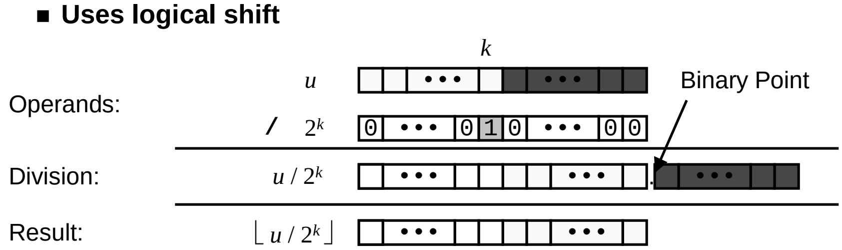
Unsigned Power-of-2 Divide with Shift

Quotient of Unsigned by Power of 2

- **u >> k gives ⌊ u / 2^k ⌋**
- **Uses logical shift**



	Division	Computed	Hex	Binary
x	15213	15213	3B 6D	00111011 01101101
x >> 1	7606.5	7606	1D B6	**0**0011101 10110110
x >> 4	950.8125	950	03 B6	**0000**0011 10110110
x >> 8	59.4257813	59	00 3B	**00000000** 00111011

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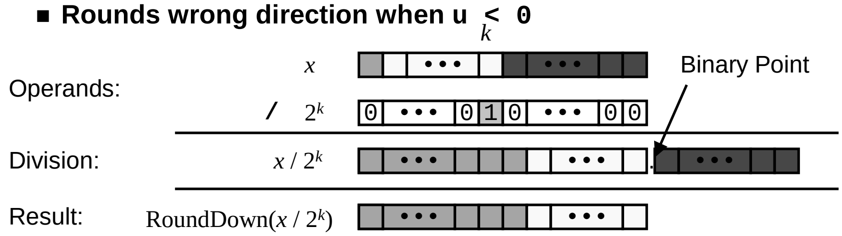
Signed Power-of-2 Divide with Shift

Quotient of Signed by Power of 2

- **`x >> k` gives $\lfloor$ `x / ` $2^k \rfloor$**
- **Uses arithmetic shift**
- **Rounds wrong direction when `u < 0`**



	Division	Computed	Hex	Binary
`y`	-15213	-15213	`C4 93`	`11000100 10010011`
`y >> 1`	-7606.5	-7607	`E2 49`	`**1**1100010 01001001`
`y >> 4`	-950.8125	-951	`FC 49`	`**1111**1100 01001001`
`y >> 8`	-59.4257813	-60	`FF C4`	`**11111111** 11000100`

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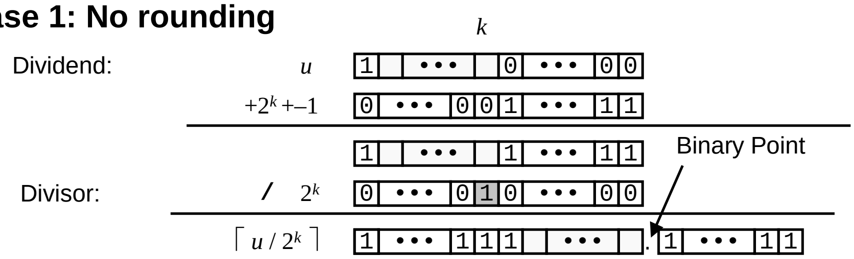
Correct Power-of-2 Divide

Quotient of Negative Number by Power of 2

- **Want $\lceil$ `x / ` $2^k \rceil$ (Round Toward 0)**
- **Compute as $\lfloor$ `(x+`2^k`-1)/ `$2^k \rfloor$**
  - **In C: `(x + (1<<k)-1) >> k`**
  - **Biases dividend toward 0**

Case 1: No rounding



Biasing has no effect

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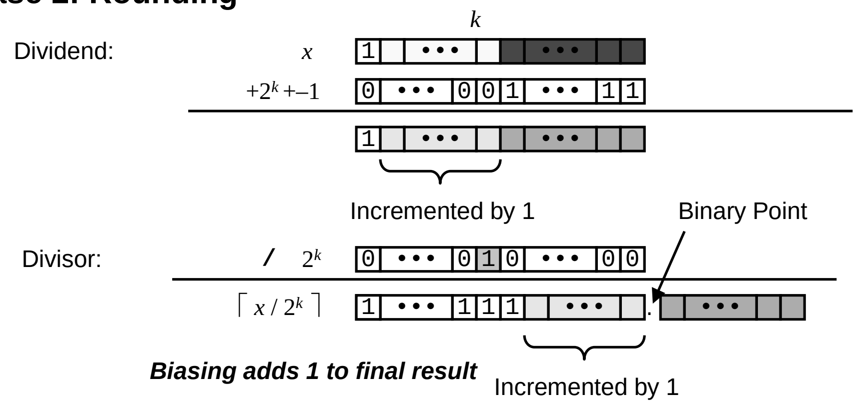
Correct Power-of-2 Divide (Cont.)

Case 2: Rounding



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Properties of Unsigned Arithmetic

Unsigned Multiplication with Addition Forms Commutative Ring

- **Addition is commutative group**
- **Closed under multiplication**
  $0 \leq \mathrm{UMult}_w(u, v) \leq 2^w - 1$
- **Multiplication Commutative**
  $\mathrm{UMult}_w(u, v) = \mathrm{UMult}_w(v, u)$
- **Multiplication is Associative**
  $\mathrm{UMult}_w(t, \mathrm{UMult}_w(u, v)) = \mathrm{UMult}_w(\mathrm{UMult}_w(t, u), v)$
- **1 is multiplicative identity**
  $\mathrm{UMult}_w(u, 1) = u$
- **Multiplication distributes over addtion**
  $\mathrm{UMult}_w(t, \mathrm{UAdd}_w(u, v)) = \mathrm{UAdd}_w(\mathrm{UMult}_w(t, u), \mathrm{UMult}_w(t, v))$

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Properties of Two's Comp. Arithmetic

Isomorphic Algebras

- **Unsigned multiplication and addition**
  - **Truncating to *w* bits**
- **Two's complement multiplication and addition**
  - **Truncating to *w* bits**

Both Form Rings

- **Isomorphic to ring of integers mod** 2^w

Comparison to Integer Arithmetic

- **Both are rings**
- **Integers obey ordering properties, e.g.,**

  $u > 0 \quad \Rightarrow \quad u + v > v$

  $u > 0, v > 0 \quad \Rightarrow \quad u \cdot v > 0$
- **These properties are not obeyed by two's comp. arithmetic**

  TMax `+ 1` `==` ***TMin***

  `15213 * 30426` `==` `-10030` (16-bit words)

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C Puzzle Answers

- **Assume machine with 32 bit word size, two's comp. integers**
- *TMin* **makes a good counterexample in many cases**

❑ **`x < 0`**	⇒	**`((x*2) < 0)`**	**False:**	*TMin*
❑ **`ux >= 0`**			**True:**	0 = *UMin*
❑ **`x & 7 == 7`**	⇒	**`(x<<30) < 0`**	**True:**	$x_1 = 1$
❑ **`ux > -1`**			**False:**	0
❑ **`x > y`**	⇒	**`-x < -y`**	**False:**	`-1`, *TMin*
❑ **`x * x >= 0`**			**False:**	`30426`
❑ **`x > 0 && y > 0`**	⇒	**`x + y > 0`**	**False:**	*TMax*, *TMax*
❑ **`x >= 0`**	⇒	**`-x <= 0`**	**True:**	–*TMax* < 0
❑ **`x <= 0`**	⇒	**`-x >= 0`**	**False:**	*TMin*

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