Recitation 6

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- Problem 2.2:31: Show that for $A$ and $B$ subsets of some universal set $U$,

$$A \subseteq B \iff \bar{B} \subseteq \bar{A}$$

$A \subseteq B \iff 
\forall x, \ x \in A \rightarrow x \in B$ \hspace{1cm} Definition of set inclusion

$\iff x \notin A \lor x \in B$ \hspace{1cm} Implication rule

$x \in B \lor x \notin A$ \hspace{1cm} Commutativity

$x \notin B \rightarrow x \notin A$ \hspace{1cm} Implication rule

$x \in \bar{B} \rightarrow x \in \bar{A}$ \hspace{1cm} Definition of set complement

$\bar{B} \subseteq \bar{A}$ \hspace{1cm} by definition of a set inclusion

QED

- 2.2.37 c: Show that if $A$ is a subset of universal set $U$

$$A \oplus U = \bar{A}$$

$\forall x, \ x \in A \oplus U \iff$

$$((x \in A) \lor (x \in U)) \land \neg((x \in A) \land (x \in U)) \iff$$

Definition of symmetric difference $\oplus$ on page 137

$$((x \in A \cup U)) \land \neg((x \in A \cap U)) \iff$$

Definition set union, intersection

$$(x \in U) \land \neg(x \in A) \iff$$

Domination, identity laws

$$\neg(x \in A) \land (x \in U) \iff$$

Commutative law (logic)

$$(x \notin A) \land (x \in U) \iff$$

Moving negation inward

$$x \in \bar{A}$$

Definition of set absolute complement

We showed that $\forall x, \ x \in A \oplus U \iff x \in \bar{A}$. Thus, $A \oplus U = \bar{A}$. \hspace{1cm} □

- Suppose that $A \cup B = \emptyset$, what can you conclude?

Answer: we conclude that $(A = \emptyset) \land (B = \emptyset)$.

We formally prove that:

$$A \cup B = \emptyset \iff (A = \emptyset) \land (B = \emptyset)$$

First we consider prove the following statement:

$$A \cup B = \emptyset \rightarrow (A = \emptyset) \land (B = \emptyset)$$
The proof is by contradiction. We assume the antecedent and negate the conclusion.

\( A \cup B = \emptyset \) \hspace{1cm} \text{given}

\( \neg((A = \emptyset) \land (B = \emptyset)) \) \hspace{1cm} \text{negating the conclusion}

\( (A \neq \emptyset) \lor (B \neq \emptyset) \) \hspace{1cm} \text{moving negation inward}

\( (A \neq \emptyset) \lor (B \neq \emptyset) \) \hspace{1cm} \text{moving negation inward}

We continue the proof using a proof by cases. Expression (4) states that, at least one of the following cases must hold and both can also hold:

(a) There is at least an element in \( A \), assume we have \( x \in A \)

(b) There is at least an element in \( B \), assume we have \( y \in B \)

Case (a) above: \( x \in A \Rightarrow x \in \{A \cup B\} \) by definition of set union \( A \cup B \neq \emptyset \), which contradicts the premise (1).

Case (b) can be shown to yield the same contradiction (exchanging \( A \) for \( B \) in the above case).

WLOG, we can conclude that

\[ A \cup B = \emptyset \Rightarrow (A = \emptyset) \land (B = \emptyset) \]

Proving the implication in the opposite direction is straightforward by definition of set union:

\[ (A = \emptyset) \land (B = \emptyset) \Rightarrow A \cup B = \emptyset \]

\( \square \)

• Now let’s look at functions, say we have the following function: \( f : \mathbb{R} \rightarrow \mathbb{R} \) where \( f(x) = \lfloor \frac{x}{2} \rfloor \)

  - First what does the graph of this function look like?
  - is \( f \) one-to-one (i.e., injective)? No, for example both 1 and 1.1 are assigned 0.
  - Is \( f \) onto \( \mathbb{R} \) (i.e., surjective)? No, the floor function only maps to integers, so only integers would be mapped to.

• Let \( A = \{1, 2, 3, 4\}, B = \{a, b, c\}, \) and \( C = \{2, 7, 10\} \)

Consider the following two functions: \( g : A \rightarrow B \) and \( f : B \rightarrow C \) where \( g : \{(1, b), (2, a), (3, a), (4, b)\} \) and \( f : \{(a, 10), (b, 7), (c, 2)\} \)

  - Find \( f \circ g \). Answer: \( \{(1, 7), (2, 10), (3, 10), (4, 7)\} \)
  - Find \( f^{-1} \). Answer: \( \{(10, a), (7, b), (2, c)\} \)
  - Is \( g^{-1} \) a function? Answer: No, because \( a \) has two pre-images but in a function, each element of the domain must be mapped to \textit{exactly one} element in the co-domain.
- Find \( f \circ f^{-1} \). Answer: \( \{(10, 10), (7, 7), (2, 2)\} \)

- Prove or disprove: \( \forall x, y \in \mathbb{R}, [x \times y] \leq [x] \times [y] \)
  - let \( x = 3.5 \), and \( y = 1.5 \). \( [3.5 \times 1.5] = [5.25] = 5 \), but \( [3.5] \times [1.5] = 4 \)
  - \( 5 \neq 4 \), therefore the statement does not hold. This is a proof with a counterexample.

- Prove or disprove for all \( x, y \in \mathbb{R}, [x \times y] \leq [x] \times [y] \)
  - Here the same example works \( [3.5 \times 1.5] = [5.25] = 6 \), but \( [3.5] \times [1.5] = 4 \times 2 = 8 \)

- Show that the function \( f : \mathbb{R} \rightarrow \mathbb{R}^+ \) where \( f(x) = |x| \) is not invertible, but if the domain is restricted to the set of nonnegative real numbers (i.e., \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \)), the resulting function is invertible.
  
  For a function to be invertible, it must be bijective (i.e., a one-to-one correspondence). Therefore, we need to check if this is one-to-one (injective) and onto (surjective).

  1. Injective: No, \( f(x_1) \neq f(x_2) \Rightarrow |x_1| \neq |x_2| \Rightarrow \pm x_1 \neq \pm x_2 \)
     
     Now, if the domain is restricted to the set of nonnegative real numbers. Is \( f(x) \) injective?
     
     \( f(x_1) = f(x_2) \Rightarrow |x_1| = |x_2| \Rightarrow x_1 = x_2 \). Therefore, on the restricted domain \( f(x) \) is injective.

  2. Surjective: Every element in codomain (\( \mathbb{R}^+ \)) is a positive number, then for \( \forall b \in \text{codomain}(f) \exists a \in \mathbb{R} \) \( b = |a| \). Thus, \( b \) has necessarily a preimage. Thus, the range and the codomain are equal, we can conclude that \( f \) is surjective.

  3. Bijective: No, because it is not injective.
     
     However, on the restricted domain, it is bijective because it is both injective and surjective.

  4. Invertible: Again, only on the restricted domain.

- Now a quick review of membership, determine whether these statements are true or false:

  1. \( \{a, b\} \subseteq \{\{a, b\}\} \)
     
     False, because neither \( a \) nor \( b \) is an element in \( \{a, b\} \).

  2. \( \{a, b\} \in \{\{a, b\}\} \)
     
     True, because there the element \( \{a, b\} \) is in \( \{\{a, b\}\} \)

  3. \( \{a, b, c\} \subset \{a, b, c\} \)
     
     False, because the sets are equal, and the statement is wondering if it is a strict subset.
4. \( \{a, b, c\} \subseteq \{a, b, c\} \)
   True, because the sets are equal.

5. \( \{\} \subseteq \{a, b, c\} \)
   True, because the empty set \( \emptyset = \{\} \) is a subset of all sets.

6. \( \emptyset \in \{a, b, c\} \)
   False, because the element \( \emptyset \) is not in the set \( \{a, b, c\} \).

7. \( \{a\} \subset \{a, a\} \)
   \textit{Trick question: watch out!}
   False, the set \( \{a, a\} \) is really \( \{a\} \) because, in a set, elements are \textit{not} repeated.
   Therefore, \( \{a\} \subset \{a, a\} \) is \textit{false} because \( \{a\} \nsubseteq \{a\} \) (Note that \( \{a\} \subseteq \{a\} \) though).