Recitation 6

Created by Taylor Spangler, Adapted by Beau Christ

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• Problem 2.2:31: Show that for \( A \) and \( B \) subsets of some universal set \( U \),

\[
A \subseteq B \iff \bar{B} \subseteq \bar{A}
\]

\[
A \subseteq B \iff \forall x, x \in A \rightarrow x \in B \quad \text{Definition of set inclusion}
\]

\[
\iff x \notin A \lor x \in B \iff \text{Implication rule}
\]

\[
x \in B \lor x \notin A \iff \text{Commutativity}
\]

\[
x \notin B \rightarrow x \notin A \iff \text{Implication rule}
\]

\[
\bar{B} \subseteq \bar{A} \quad \text{by definition of a set inclusion}
\]

QED

• 2.2.37 c: Show that if \( A \) is a subset of universal set \( U \)

\[
A \oplus U = \bar{A}
\]

\[
\forall x, x \in A \oplus U \iff ((x \in A) \lor (x \in U)) \land \neg((x \in A) \land (x \in U)) \iff \quad \text{definition of symmetric difference \( \oplus \) on page 137}
\]

\[
((x \in A \cup U)) \land \neg((x \in A \cap U)) \iff \quad \text{Definition set union, intersection}
\]

\[
(x \in U) \land \neg(x \in A) \iff \quad \text{Domination, identity laws}
\]

\[
\neg(x \in A) \land (x \in U) \iff \quad \text{Commutative law (logic)}
\]

\[
(x \notin A) \land (x \in U) \iff \quad \text{Moving negation inward}
\]

\[
x \in \bar{A} \quad \text{Definition of set absolute complement}
\]

We showed that \( \forall x, x \in A \oplus U \iff x \in \bar{A} \). Thus, \( A \oplus U = \bar{A} \). \( \square \)

• Suppose that \( A \cup B = \emptyset \), what can you conclude?

Answer: we conclude that \( (A = \emptyset) \land (B = \emptyset) \).

We formally prove that:

\[
A \cup B = \emptyset \iff (A = \emptyset) \land (B = \emptyset)
\]

First we consider prove the following statement:

\[
A \cup B = \emptyset \rightarrow (A = \emptyset) \land (B = \emptyset)
\]
The proof is by contradiction. We assume the antecedent and negate the conclusion.

(1) \( A \cup B = \emptyset \) given
(2) \( \neg((A = \emptyset) \land (B = \emptyset)) \) negating the conclusion
(3) \( (A \neq \emptyset) \lor (B \neq \emptyset) \) moving negation inward
(4) \( (A \neq \emptyset) \lor (B \neq \emptyset) \) moving negation inward

We continue the proof using a proof by cases. Expression (4) states that, at least one of the following cases must hold and both can also hold:

(a) There is at least an element in \( A \), assume we have \( x \in A \)
(b) There is at least an element in \( B \), assume we have \( y \in B \)

Case (a) above: \( x \in A \Rightarrow x \in \{A \cup B\} \) by definition of set union \( A \cup B \neq \emptyset \), which contradicts the premise (1).

Case (b) can be shown to yield the same contradiction (exchanging \( A \) for \( B \) in the above case).

WLOG, we can conclude that

\[
A \cup B = \emptyset \Rightarrow (A = \emptyset) \land (B = \emptyset)
\]

Proving the implication in the opposite direction is straightforward by definition of set union:

\[
(A = \emptyset) \land (B = \emptyset) \Rightarrow A \cup B = \emptyset
\]

□

• Now let’s look at functions, say we have the following function: \( f : \mathbb{R} \to \mathbb{R} \) where \( f(x) = \lfloor \frac{x}{2} \rfloor \)
  - First what does the graph of this function look like?
  - is \( f \) one-to-one (i.e., injective)? No, for example both 1 and 1.1 are assigned 0.
  - Is \( f \) onto \( \mathbb{R} \) (i.e., surjective)? No, the floor function only maps to integers, so only integers would be mapped to.

• Let \( A = \{1, 2, 3, 4\} \), \( B = \{a, b, c\} \), and \( C = \{2, 7, 10\} \)

Consider the following two functions: \( g : A \to B \) and \( f : B \to C \) where \( g : \{(1, b), (2, a), (3, a), (4, b)\} \) and \( f : \{(a, 10), (b, 7), (c, 2)\} \)

- Find \( f \circ g \). Answer: \( \{(1, 7), (2, 10), (3, 10), (4, 7)\} \)
- Find \( f^{-1} \). Answer: \( \{(10, a), (7, b), (2, c)\} \)
- Is \( g^{-1} \) a function? Answer: No, because \( a \) has two pre-images but in a function, each element of the domain must be mapped to exactly one element in the co-domain.
• Find \( f \circ f^{-1} \). Answer: \( \{(10, 10), (7, 7), (2, 2)\} \)

• Prove or disprove: \( \forall x, y \in \mathbb{R}, [x \times y] \leq [x] \times [y] \)
  - let \( x = 3.5 \), and \( y = 1.5 \). \([3.5 \times 1.5] = [5.25] = 5\), but \([3.5] \times [1.5] = 4\)
  - \( 5 \neq 4 \), therefore the statement does not hold. This is a proof with a counterexample.

• Prove or disprove for all \( x, y \in \mathbb{R}, [x \times y] \leq [x] \times [y] \)
  - Here the same example works \([3.5 \times 1.5] = [5.25] = 6\), but \([3.5] \times [1.5] = 4 \times 2 = 8\)

• Show that the function \( f : \mathbb{R} \to \mathbb{R}^+ \) where \( f(x) = |x| \) is not invertible, but if the domain is restricted to the set of nonnegative real numbers (i.e., \( f : \mathbb{R}^+ \to \mathbb{R}^+ \)), the resulting function is invertible.
  
  For a function to be invertible, it must be bijective (i.e., a one-to-one correspondence). Therefore, we need to check if this is one-to-one (injective) and onto (surjective).

  1. Injective: No, \( f(x_1) \neq f(x_2) \Rightarrow |x_1| \neq |x_2| \Rightarrow \pm x_1 \neq \pm x_2 \)

     Now, if the domain is restricted to the set of nonnegative real numbers. Is \( f(x) \) injective?

     \( f(x_1) = f(x_2) \Rightarrow |x_1| = |x_2| \Rightarrow x_1 = x_2 \). Therefore, on the restricted domain \( f(x) \) is injective.

  2. Surjective: Every element in codomain (\( \mathbb{R}^+ \)) is a positive number, then for \( \forall b \in \text{codomain}(f) \exists a \in \mathbb{R} : b = |a| \). Thus, \( b \) has necessarily a preimage. Thus, the range and the codomain are equal, we can conclude that \( f \) is surjective.

  3. Bijective: No, because it is not injective.

     However, on the restricted domain, it is bijective because it is both injective and surjective.

  4. Invertible: Again, only on the restricted domain.

• Now a quick review of membership, determine whether these statements are true or false:

  1. \( \{a, b\} \subseteq \{\{a, b\}\} \)
     
     False, because neither \( a \) nor \( b \) is an element in \( \{\{a, b\}\} \).

  2. \( \{a, b\} \in \{\{a, b\}\} \)
     
     True, because there the element \( \{a, b\} \) is in \( \{\{a, b\}\} \)

  3. \( \{a, b, c\} \subset \{a, b, c\} \)
     
     False, because the sets are equal, and the statement is wondering if it is a strict subset.
4. \( \{a, b, c\} \subseteq \{a, b, c\} \)
   True, because the sets are equal.

5. \( \{\} \subseteq \{a, b, c\} \)
   True, because the empty set \( \emptyset = \{\} \) is a subset of all sets.

6. \( \emptyset \in \{a, b, c\} \)
   False, because the element \( \emptyset \) is not in the set \( \{a, b, c\} \).

7. \( \{a\} \subset \{a, a\} \)
   Trick question: watch out!
   False, the set \( \{a, a\} \) is really \( \{a\} \) because, in a set, elements are not repeated.
   Therefore, \( \{a\} \subset \{a, a\} \) is false because \( \{a\} \not\subset \{a\} \) (Note that \( \{a\} \subseteq \{a\} \) though).