Recitation 5

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• To start, we’ll look at some proofs. Consider problem 1.7.19:
  – Let $P(n)$: If $a$ and $b$ are positive real numbers, then $(a + b)^n \geq a^n + b^n$.
  – Prove that $P(1)$ is true.
    1. First, let $a$ and $b$ be positive real numbers
    2. $P(1): (a + b)^1 \geq a^1 + b^1$.
    4. Well this is clearly true, since $a + b = a + b$.
  – What kind of proof did we use? This is a direct proof.

• Now a more difficult proof 1.7.29:
  – Prove or disprove:
    \[ \forall m, n \in \mathbb{Z}, \ mn = 1 \Rightarrow (m = 1 \land n = 1) \lor (m = -1 \land n = -1) \]
    1. Case 1: Suppose $|m| > 1$. Regardless of what $n$ is, $mn = 1$, so $n = \frac{1}{|m|}$. But we said that both $m$ and $n$ are integers. If $|m| > 1$, then $n$ here cannot be an integer. So, contradiction.
    2. Case 2: Similar to case 1, but suppose $|n| > 1$. Completely symmetric, works out the same.
    3. Case 3: Suppose $m = 1$ and $n = -1$. Clearly then $mn = -1$. So, contradiction.
    5. Case 5: Suppose $m = 1$ and $n = 1$. Then $mn = 1$.
    6. Case 6: Suppose $m = 1$ and $n = 1$. Then $mn = 1$.
  – What techniques did we use? We used proofs by cases and by contradiction

• 1.8: Example 10: Show that there is a positive integer that can be written as the sum of cubes of positive integers, in two different ways:
– Here, we are going to use a constructive existence proof.
– $1729 = 10^3 + 9^3 = 12^3 + 1^3$.
– So, 1729 can be written of the sum of cubes of positive integers in two different ways.
– Therefore, there is a positive integer that can be written as the sum of cubes of positive integers, in two different ways.
– These types of proof are basically finding an example.
– Unfortunately, other than using your intuition, the only way to go about it is brute force.

• 1.8: Example 20: Can we tile a standard chess board with opposite corners removed using dominos (e.g., upper left, lower right removed)?
  – Suppose we can.
  – First, note that a standard chessboard can be covered by dominos, each having one white and one black square.
  – We know that the standard chess board has 64 square. Removing 2 gives $64 - 2 = 62$ squares.
  – Tiling with dominos uses $62 / 2 = 31$ dominos.
  – These dominos each have one white and one black square (like chessboards have white and black).
  – So, using the tiles we have 31 white and 31 black squares.
  – However, when we remove two opposite corner squares, either 32 of the remaining squares are white and 30 are black, or 32 are black and 30 are white.
  – But, we said we had 31 of each color. Thus, we have a contradiction.
  – Therefore, we cannot tile a standard chessboard with two opposite corners removed, with dominos.
• Now moving on to sets, 2.1.11: Determine whether true or false:

a. \( x \in \{x\} \) is true.
b. \( \{x\} \subseteq \{x\} \) is true.
c. \( \{x\} \in \{x\} \) is false.
d. \( \{x\} \in \{\{x\}\} \) is true.
e. \( \emptyset \subseteq \{x\} \) is true.
f. \( \emptyset \in \{x\} \) is false.

• 2.1.27(a) Let \( A = \{a, b, c, d\} \), and \( B = \{y, z\} \).

a. \( A \times B \)?
   1. \( |A \times B| \)? As in, how big will it be? \( 4 \times 2 = 8 \)
   2. \( \{(a, y), (b, y), (c, y), (d, y), (a, z), (b, z), (c, z), (d, z)\} \)

   – How about \( \mathbb{Z} \times (\mathbb{Z}^+ \setminus \emptyset) \)
     1. What is its cardinality? \( \infty \)
     2. What are some of its elements? \((1,2), (3,4), (-1,5)\).
     3. Is \( (2, 0) \in \mathbb{Z} \times (\mathbb{Z}^+ \setminus \emptyset) \)? No
     4. Does this set remind you of anything? Could be used to represent the rational numbers, \( \mathbb{Q} \).

• Computer the power set of the following set \( S = \{a, b, \{c\}, \emptyset\} \)

   – First, how many elements are in \( \mathcal{P}(S) \)? \( 2^4 = 16 \)

   – \( \{\emptyset, \{\emptyset\}, \{\{c\}\}, \{a\}, \{b\}, \{a, b\}, \{a, \{c\}\}, \{b, \{c\}\}, \{a, \emptyset\}, \{b, \emptyset\}, \{\{c\}, \emptyset\}, \{a, b, \{c\}\}, \{a, b, \emptyset\}, \{b, \{c\}, \emptyset\}, \{a, \{c\}, \emptyset\}, \{a, b, \{c\}, \emptyset\}\} \)

• (Last 15 minutes) Give quiz