To start, we’ll look at some proofs. Consider problem 1.7.19:

- Let $P(n)$: If $a$ and $b$ are positive real numbers, then $(a+b)^n \geq a^n + b^n$.
- Prove that $P(1)$ is true.
  1. First, let $a$ and $b$ be positive real numbers
  2. $P(1): (a + b)^1 \geq a^1 + b^1$.
  4. Well this is clearly true, since $a + b = a + b$.
- What kind of proof did we use? This is a direct proof.

Now a more difficult proof 1.7.29:

- Prove or disprove:
  \[ \forall m, n \in \mathbb{Z}, mn = 1 \Rightarrow (m = 1 \land n = 1) \lor (m = -1 \land n = -1) \]
  1. Case 1: Suppose $|m| > 1$. Regardless of what $n$ is, $mn = 1$, so $n = \frac{1}{|m|}$. But we said that both $m$ and $n$ are integers. If $|m| > 1$, then $n$ here cannot be an integer. So, contradiction.
  2. Case 2: Similar to case 1, but suppose $|n| > 1$. Completely symmetric, works out the same.
  3. Case 3: Suppose $m = 1$ and $n = -1$. Clearly then $mn = -1$. So, contradiction.
  5. Case 5: Suppose $m = 1$ and $n = 1$. Then $mn = 1$.
  6. Case 6: Suppose $m = 1$ and $n = 1$. Then $mn = 1$.
- What techniques did we use? We used proofs by cases and by contradiction.

1.8: Example 10: Show that there is a positive integer that can be written as the sum of cubes of positive integers, in two different ways:
Here, we are going to use a *constructive existence* proof.

\[ 1729 = 10^3 + 9^3 = 12^3 + 1^3. \]

So, 1729 can be written of the sum of cubes of positive integers in two different ways.

Therefore, there is a positive integer that can be written as the sum of cubes of positive integers, in two different ways.

These types of proof are basically finding an example.

Unfortunately, other than using your intuition, the only way to go about it is brute force.

1.8: Example 20: Can we tile a standard chess board with opposite corners removed using dominos (e.g., upper left, lower right removed)?

Suppose we can.

First, note that a standard chessboard can be covered by dominos, each having one white and one black square.

We know that the standard chess board has 64 square. Removing 2 gives 64-2 = 62 squares.

Tiling with dominos uses 62/2 = 31 dominos.

These dominos each have one white and one black square (like chessboards have white and black).

So, using the tiles we have 31 white and 31 black squares.

However, when we remove two opposite corner squares, either 32 of the remaining squares are white and 30 are black, or 32 are black and 30 are white.

But, we said we had 31 of each color. Thus, we have a contradiction.

Therefore, we cannot tile a standard chessboard with two opposite corners removed, with dominos.
• Now moving on to sets, 2.1.11: Determine whether true or false:

a. $x \in \{x\}$ is true.
b. $\{x\} \subseteq \{x\}$ is true.
c. $\{x\} \in \{x\}$ is false.
d. $\{x\} \in \{\{x\}\}$ is true.
e. $\emptyset \subseteq \{x\}$ is true.
f. $\emptyset \in \{x\}$ is false.

• 2.1.27(a) Let $A = \{a, b, c, d\}$, and $B = \{y, z\}$.

a. $A \times B$?
   1. $|A \times B|$? As in, how big will it be? $4 \times 2 = 8$
   2. $\{(a, y), (b, y), (c, y), (d, y), (a, z), (b, z), (c, z), (d, z)\}$
   - How about $\mathbb{Z} \times (\mathbb{Z}^+ \setminus 0)$
     1. What is its cardinality? $\infty$
     2. What are some of its elements? $(1,2),(3,4),(-1,5)$.
     3. Is $(2,0) \in \mathbb{Z} \times (\mathbb{Z}^+ \setminus 0)$? No
     4. Does this set remind you of anything? Could be used to represent the rational numbers, $\mathbb{Q}$.

• Computer the power set of the following set $S = \{a, b, \{c\}, \emptyset\}$

   - First, how many elements are in $\mathcal{P}(S)$? $2^4 = 16$.
   - $\emptyset$, $\{\emptyset\}$, $\{\{c\}\}$, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{a, \{c\}\}$, $\{b, \{c\}\}$, $\{a, \emptyset\}$, $\{b, \emptyset\}$, $\{\{c\}, \emptyset\}$, $\{a, b, \{c\}\}$, $\{a, b, \emptyset\}$, $\{b, \{c\}\}$, $\{a, \{c\}\}$, $\{a, \{c\}, \emptyset\}$, $\{a, b, \{c\}, \emptyset\}$