• 7.1.9d) Use backwards substitution to solve $a_n = a_{n-1} + 2n + 3$, when $a_0 = 4$.

\[
\begin{align*}
  a_n &= a_{n-1} + 2n + 3 \\
  &= (a_{n-2} + 2(n - 1) + 3) + 2n + 3 \\
  &= a_{n-2} + 2n - 2 + 3 + 2n + 3 \\
  &= a_{n-2} + 4n + 2(3) - 2 \\
  &= (a_{n-3} + 2(n - 2) + 3) + 4n + 2(3) - 2 \\
  &= a_{n-3} + 2n - 4 + 3 + 4n + 2(3) - 2 \\
  &= a_{n-3} + 6n + 3(3) - 2 - 4 \\
  &= (a_{n-4} + 2(n - 3) + 3) + 6n + 3(3) - 2 - 4 \\
  &= a_{n-4} + 2n - 6 + 3 + 6n + 3(3) - 2 - 4 \\
  &= a_{n-4} + 8n + 4(3) - 2 - 4 - 6 \\
  &= \ldots \\
  &= a_{n-n} + 2n(n) + n(3) - 2 - 4 - 6 - \ldots - 2(n - 1) \\
  &= a_0 + 2n^2 + 3n - \sum_{i=1}^{n-1} 2i \\
  &= a_0 + 2n^2 + 3n - 2\frac{(n - 1)(n - 1 + 1)}{2} \\
  &= a_0 + 2n^2 + 3n - n(n - 1) \\
  &= a_0 + 2n^2 + 3n - n^2 + n \\
  &= n^2 + 4n + 4 \\
\end{align*}
\]

• Find the solution to the recurrence relation:

\[
\begin{align*}
  a_n &= 2a_{n-1} + 8a_{n-2} \\
  a_0 &= 3 \\
  a_1 &= 4
\end{align*}
\]
This relation is a linear homogeneous recurrence relation of degree 2 because:

- The right-hand side has only multiples of previous terms of the sequence and coefficients are all constants. Therefore, it is linear.
- No terms occur that are not multiples of \( a_j \) (i.e., no non-recursive cost). Therefore, it is homogeneous.
- It is expressed in terms of the \((n - 2)\)th term of the sequence. Therefore, it is of degree 2).

We know that a solution to solve this recurrence relation is of the form \( a_n = r^n \) where \( r \) is some real constant. Replacing the solution in the recurrence relation, we get:

\[
r^n = 2r^{n-1} + 8r^{n-2}.
\]

Dividing by \( r^{n-2} \), we get:

\[
r^2 = 2r + 8.
\]

Thus, the \textit{characteristic equation} of this recurrence relation is:

\[
r^2 - 2r - 8 = (r + 2)(r - 4) = 0.
\]

This characteristic equation has the roots \( r_1 = -2 \) and \( r_2 = 4 \); Therefore, the solution of the recurrence relation is

\[
a_n = \alpha_1 (-2)^n + \alpha_2 4^n.
\]

Plugging in our initial conditions we get

\[
3 = \alpha_1 + \alpha_2
\]

\[
4 = -2\alpha_1 + 4\alpha_2
\]

Solving for \( \alpha_1 = 3 - \alpha_2 \), we get \( 4 = -2(3 - \alpha_2) + 4\alpha_2 \Rightarrow 4 = -6 + 2\alpha_2 + 4\alpha_2 \Rightarrow \frac{5}{3} = \alpha_2 \). Therefore, \( \alpha_1 = \frac{4}{3} \) and \( \alpha_2 = \frac{5}{3} \).

Putting the values of \( \alpha_1, \alpha_2 \) back in the solution form, we obtain the following solution of the recurrence relation given the boundary conditions

\[
a_n = \frac{4}{3} (-2)^n + \frac{5}{3} 4^n.
\]

\[
\textit{You are not responsible for solving non-homogeneous recurrence relations.}
\]

Solve the following linear non-homogeneous recurrence relation:

\[
a_n = 2a_{n-1} - 8a_{n-2} + n \quad \text{(1)}
\]

\[
a_0 = 3 \quad \text{(2)}
\]

\[
a_1 = 4 \quad \text{(3)}
\]
We notice that \( f(n) \) is polynomial \( n \). We will solve this problem using Theorem 6 on page 469, which covers this case, the case that \( f(n) \) is an exponential in \( n \), and the case where \( f(n) \) is a product of a polynomial and an exponential in \( n \).

First, we solve the associated linear homogeneous recurrence relation, which happens to be the one above :-). Its solution is:

\[
a_n = \alpha_1 (-2)^n + \alpha_2 4^n.
\]

Next, we find a particular solution for the given non-homogeneous term. Theorem 6 applies to \( f(n) \) of the form:

\[
f(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0) s^n.
\]

In our case, \( f(n) = n \) and \( s \) is 1. Since our \( s \) is \textit{not} a root of our characteristic equation (*relief*), there is a particular solution of the form:

\[
a^p = (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) s^n.
\]

For us, the particular solution is \( a^p = p_1 n + p_0 \). Plugging the particular solution in Equation (1), we get:

\[
2p_1 n - 2p_1 + 2p_0 - 8p_1 n - 16p_1 - 8p_0 + n = 0
\]

Moving all terms to one side of the equation, we get:

\[
(7p_1 - 1)n + (-14p_1 + 7p_0) = 0.
\]

Given that \( n \neq 0 \), we must have the following:

\[
7p_1 - 1 = 0
\]

\[
-14p_1 + 7p_0 = 0
\]

Now, \( 7p_1 - 1 = 0 \) \( \Rightarrow \) \( 7p_1 = 1 \) \( \Rightarrow \) \( p_1 = \frac{1}{7} \).

Further, \(-14p_1 + 7p_0 = 0 \) \( \Rightarrow \) \(-14(\frac{1}{7}) + 7p_0 = 0 \) \( \Rightarrow \) \(-2 + 7p_0 = 0 \) \( \Rightarrow \) \( 7p_0 = 2 \) \( \Rightarrow \) \( p_0 = \frac{2}{7} \).

We have thus found the particular solution: have \( a^p = \frac{1}{7} n + \frac{2}{7} \). Therefore,

\[
a_n = a^h + a^p
\]

\[
a_n = (\alpha_1 (-2)^n + \alpha_2 4^n) + \left( \frac{1}{7} n + \frac{2}{7} \right).
\]

To determine the values of \( \alpha_1, \alpha_2 \), we plug in our initial conditions:

\[
3 = \frac{2}{7} + \alpha_1 + \alpha_2
\]

\[
4 = \frac{1}{7} + \frac{2}{7} - 2\alpha_1 + 4\alpha_2
\]

\[
3
\]
Or:
\[
\begin{align*}
\frac{19}{7} &= \alpha_1 + \alpha_2 \\
\frac{25}{7} &= -2\alpha_1 + 4\alpha_2
\end{align*}
\]
Solving for \(\alpha_1 = \frac{19}{7} - \alpha_2\), we get \(\frac{25}{7} = -2(\frac{19}{7} - \alpha_2) + 4\alpha_2 \Rightarrow \frac{25}{7} = -\frac{38}{7} + 2\alpha_2 + 4\alpha_2 \Rightarrow \frac{63}{7} = 6\alpha_2 \Rightarrow \frac{63}{42} = \alpha_2\). Replacing \(\alpha_2\) by its value, we get \(\alpha_1 = \frac{-13}{42}\).

Replacing \(\alpha_1, \alpha_2\), we get
\[
a_n = \frac{1}{7}n + \frac{2}{7} + \frac{63}{42}(-2)^n + \frac{-13}{42}4^n.
\]

• Give the asymptotic characterization for \(T(n) = 3T(n/4) + 8n^3\).
Remember Master Theorem, when we have \(T(n) = aT(n/b) + f(n)\) where:
- \(T(n)\) is monotone
- \(f(n) \in \Theta(n^d)\) where \(d \geq 0\),
- \(b\) is a constant
we can use it to classify our recurrence relation as follows:
\[
T(n) \text{ is } \begin{cases} 
\Theta(n^d) & a < b^d \\
\Theta(n^{d \log_b a}) & a = b^d \\
\Theta(n^{d \log_b a}) & a > b^d
\end{cases}
\]
Therefore, we can use Master’s theorem and \(a = 3\), \(b = 4\) and \(d = 3\). Therefore, \(T(n)\) is \(\Theta(n^3)\) because \(a < b^d (3 < 64)\).