Induction: Example using triominoes for $2^n \times 2^n$ checkerboard missing one corner, see page 326.

Problem 5.1.5: Using induction, prove:

$$\forall n \geq 0 \quad 1^2 + 3^2 + \ldots + (2n + 1)^2 = \frac{(n + 1)(2n + 1)(2n + 3)}{3}$$

We prove the property using mathematical induction:

1. Let’s state the property to prove:

$$P(n) : 1^2 + 3^2 + \ldots + (2n + 1)^2 = \frac{(n + 1)(2n + 1)(2n + 3)}{3}$$

2. We take $n_0 = 0$ and prove $P(0)$:

$$P(0) = 1 = \frac{(1)(1)(3)}{3}$$

We show that $P(0)$ holds: $1 = 1 \cdot 1 \cdot 3 = 1$. Therefore, $P(0)$ is true.

3. What is the inductive hypothesis?

$$P(k) : 1^2 + 3^2 + \ldots + (2k + 1)^2 = \frac{(k + 1)(2k + 1)(2k + 3)}{3}$$

4. Assuming the base case and the inductive hypothesis, we want to prove:

$$P(k + 1) : 1^2 + 3^2 + \ldots + (2k + 3)^2 = \frac{(k + 2)(2k + 3)(2k + 5)}{3}$$

5. We start from $1^2 + 3^2 + \ldots + (2k + 1)^2 + (2k + 3)^2$.

$$1^2 + 3^2 + \ldots + (2k + 1)^2 + (2k + 3)^2 = \frac{(k + 1)(2k + 1)(2k + 3)}{3} + (2k + 3)^2$$

$$= \frac{(k + 1)(2k + 1)(2k + 3)}{3} + (2k + 3)^2$$

$$= (2k + 3) \cdot \left( \frac{(k + 1)(2k + 1) + 6k + 9}{3} \right)$$

$$= (2k + 3) \cdot \left( \frac{2k^2 + 9k + 10}{3} \right)$$

$$= \frac{(k + 2)(2k + 3)(2k + 5)}{3}$$

Thus, $P(k + 1)$ holds.
Consequently, by the PMI, $\forall n \geq 0 \ 1^2 + 3^2 + \ldots + (2n + 1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$

- Now, prove the following: $3 \mid 2^{2n} - 1$ for $n \geq 1$.

1. First, we state the property to prove: $P(n) : 3 \mid 2^{2n} - 1$.

2. Base case is $n_0 = 1$. So $P(1) = 3 \mid 2^2 - 1$. Clearly, $2^2 - 1 = 4 - 1 = 3$, $3 \mid 3$, which is obvious. So, $P(1)$ holds.

3. Next, we state the inductive hypothesis: $P(k) : 3 \mid 2^k - 1$ and assume that $P(k)$ holds.

4. Now, we have to prove that $P(k+1) : 3 \mid 2^{2k+2} - 1$ holds. We start from $2^{2k+2} - 1$.
   $2^{2k+2} - 1 = 4 \cdot 2^k - 1 = 4 \cdot (2^k - 1 + 1) - 1$.
   The inductive hypothesis gives that $3 \mid 2^k - 1$ This, there is an integer $t$ such that $2^k - 1 = 3t$. Thus,
   $2^{2k+2} - 1 = 4 \cdot (3t + 1) - 1 = 12t + 4 - 1 = 12t + 3 = 3 \cdot (4t + 1)$.
   Thus, $2^{2k+2} - 1$ is a multiple of 3. Hence, $P(k+1)$ holds.

Consequently, by the PMI, $\forall n \geq 1, \ 3 \mid 2^{2n} - 1$.

- A proof by strong induction.

Show that $\forall n \in \mathbb{N}, \ 12 \mid (n^4 - n^2)$.

1. First, we state the property:

   $P(n) : 12 \mid (n^4 - n^2)$

2. Base Case:

   (a) For $n = 1$: $1^4 - 1^2 = 0 = 12 \cdot 0$, so $P(1)$ is true.
   (b) For $n = 2$: $2^4 - 2^2 = 16 - 4 = 12 = 12 \cdot 1$, so $P(2)$ is true.
   (c) For $n = 3$: $3^4 - 3^2 = 81 - 9 = 72 = 12 \cdot 6$, so $P(3)$ is true.
   (d) For $n = 4$: $4^4 - 4^2 = 256 - 16 = 240 = 12 \cdot 20$, so $P(4)$ is true.
   (e) For $n = 5$: $5^4 - 5^2 = 625 - 25 = 600 = 12 \cdot 50$, so $P(5)$ is true.
   (f) For $n = 6$: $6^4 - 6^2 = 1296 - 36 = 1260 = 12 \cdot 105$, so $P(6)$ is true.

3. Strong Inductive Hypothesis Let $k \geq 6 \in \mathbb{N}$ and assume that $12 \mid (m^4 - m^2)$ for $1 \leq m < k$ where $m \in \mathbb{N}$.

4. We need to establish $P(k)$.
   Let $i = k - 5$. Because $i < k$, we can assume that $P(i)$ holds.
   Clearly $i + 6 = k + 1$.
   $(i + 6)^4 - (i + 6)^2$
   $= (i^4 + 24i^3 + 180i^2 + 864i + 1296) - (i^2 + 12i + 36)$
   $= (i^4 - i^2) + 24i^3 + 180i^2 + 852i + 1260$
   Because $P(i)$ holds, we have $i^4 - i^2 = 12 \cdot t$. 

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Further, $24i^3 + 180i^2 + 852i + 1260 = 12(2i^3 + 15i^2 + 71i + 105)$.
Thus, $(i + 6)^4 - (i + 6)^2 = 12 \cdot t + 12(2i^3 + 15i^2 + 71i + 105) = 12(t + 2i^3 + 15i^2 + 71i + 105)$.
Hence, $(i + 6)^4 - (i + 6)^2$ is a multiple of 12 and $12 \mid (k + 1)^4 - (k + 1)^2$.

We can finally state that by the principle of strong induction, $\forall n \in \mathbb{N}, 12 \mid (n^4 - n^2)$. 